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CONDORCET PROPORTIONS AND KELLY'S CONJECTURES[1]

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Let $C(m, n)$ be the proportion of all n-tuples of linear orders on a set of m alternatives such that some alternative x is ranked ahead of y in at least $\frac{1}{2}n$ of the orders, for each $y \neq x$. Kelly proved that $C(m, n) < C(m, n+1)$ for $m \geq 3$ and odd $n \geq 3$, and that $C(m, n) > C(m, n+1)$ for $m \geq 3$ and even $n \geq 2$. He also conjectured that $C(m, n) > C(m+1, n)$ for $m \geq 3$ and $n = 3$ or $n \geq 5$, and that $C(m, n) > C(m, n+2)$ for $m \geq 3$ and $n = 1$ or $n \geq 3$. The first of these conjectures is shown to be true for $n = 3$, and for $m = 3$ and odd n. The second conjecture is established for $m \in \{3, 4\}$ and odd n, and for $m = 3$ and all large even n.

1. Introduction

A *profile* is a finite list of linear orders on a finite set of alternatives. A profile is a *Condorcet profile* if and only if it has a *Condorcet alternative*, which is an alternative x such that, for every alternative $y \neq x$, at least as many orders in the list have x ranked ahead of y as have y ranked ahead of x. In other words, a Condorcet alternative is a simple majority alternative. The *Condorcet proportion* $C(m, n)$ for n-term profiles on m alternatives is the number of n-term Condorcet profiles on m alternatives divided by $(m!)^n$, the total number of n-term profiles on m alternatives.

Most of what is presently known about Condorcet proportions can be found in Garman and Kamien [5], Niemi and Weisberg [12], DeMeyer and Plott [3], May [11], Kelly [10], and Gehrlein and Fishburn [6, 7]. All but one of these deal with precise computations and approximations of $C(m, n)$. The exception is Kelly, who examined trends in $C(m, n)$ and showed that

$$C(m, n) < C(m, n+1) \quad \text{for } m \geq 3 \text{ and odd } n \geq 3,$$
$$C(m, n) > C(m, n+1) \quad \text{for } m \geq 3 \text{ and even } n \geq 2.$$

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Kelly also proposed:

Conjecture 1. $C(m, n) > C(m+1, n)$ *for* $m \geqslant 3$, $n = 3$ *or* $n \geqslant 5$.

Conjecture 2. $C(m, n) > C(m, n+2)$ *for* $m \geqslant 3$, $n = 1$ *or* $n \geqslant 3$.

He noted that "proofs of these two conjectures would be important contributions to the formal voting literature" and discussed some of their difficulties. With n as the number of voters, Conjecture 1 says that, for fixed $n \in \{3, 5, 6, \ldots\}$, the Condorcet proportions decrease in $m \geqslant 3$; Conjecture 2 says that, for fixed $m \in \{3, 4, \ldots\}$, the Condorcet proportion for n voters exceeds the proportion for $n+2$ voters, except when $n = 2$ since $C(3, 2) = C(3, 4) = 1$.

The main purpose of this paper is to examine Conjecture 1 for $n = 3$ and Conjecture 2 for $m = 3$. Section 2 will prove that Conjecture 1 is true for three voters. Section 3 proves that Conjecture 2 is true for three alternatives and odd numbers of voters, and Section 5 notes that Conjecture 2 is true for three alternatives and almost all even numbers of voters $n \geqslant 4$. In particular, for $m = 3$ and n even, we shall prove that Conjecture 2 is true for all even n greater than some finite N. Several ancilliary results for $m = 3$ and n even will be noted in Section 4. Finally, Section 6 comments on the conjectures for odd n and larger values of m.

Although we deal only with several basic cases, the proofs tend to be involved and consume most of the paper. Our experience with these cases suggests that the more general cases will be extremely difficult to resolve. Nevertheless, we would encourage others to try them using either extensions of the proof techniques used here or entirely new techniques that we cannot now foresee.

The proof in the next section is based on the proportions formation described above. Thereafter we shall turn to probabilistic proof techniques in which $C(m, n)$ is interpreted as the probability that there will be a Condorcet alternative for the m-alternative, n-voter situation in which each voter independently selects one of the $m!$ linear orders on the alternatives as his preference order according to the equally-likely probability distribution that assigns probability $1/m!$ to each of the $m!$ linear orders.

2. Conjecture 1 for $n = 3$

All profiles in this section will be 3-term, or 3-voter, profiles. We shall prove that $C(m, 3)$ decreases in $m \geqslant 2$, where clearly $C(2, 3) = 1$.

Theorem 1. $C(m, 3) > C(m+1, 3)$ *for all* $m \in \{2, 3, \ldots\}$.

Let $B(m)$ be the number of Condorcet profiles on m alternatives that have a fixed alternative x as the necessarily unique Condorcet alternative. Since

$C(m, 3) = mB(m)/(m!)^3$, the inequality in Theorem 1 reduces to $(m+1)^2 mB(m) > B(m+1)$ for $m \geq 2$, or

$$(m+2)^2(m+1)B(m+1) > B(m+2) \quad \text{for } m \geq 1. \tag{1}$$

We shall approach (1) through a series of four lemmas. The following index set will be used in the first two lemmas and at the conclusion of the section:

$$A_m = \{(a_1, a_2, a_3) \in \{0, 1, 2, \ldots\}^3 : a_1 + a_2 + a_3 \leq m\}.$$

Lemma 1. $B(m+1) = \sum_{A_m} m!(m-a_1)!(m-a_2)!(m-a_3)!/(m-a_1-a_2-a_3)!$

Proof. For fixed x in a set of $m+1$ alternatives let a_i be the number of alternatives in the set that voter i prefers to (ranks ahead of) x, for $i = 1, 2, 3$. Then x can be the Condorcet alternative for a profile only if $(a_1, a_2, a_3) \in A_m$. Given $(a_1, a_2, a_3) \in A_m$, there are $a_i!(m-a_i)!$ linear orders on the alternatives in which exactly a_i alternatives are ranked ahead of x, and there are $m!/[a_1!a_2!a_3!(m-a_1-a_2-a_3)!]$ distinct ways that the three ahead-of-x sets with a_i members can be chosen without duplication, which is precisely what is needed for x to be the Condorcet alternative. Therefore $B(m+1)$ equals the latter expression times the product of the three $a_i!(m-a_i)!$, summed over A_m. □

For any m and (a_1, a_2, a_3) let

$$\begin{aligned} F_{m+1}(a_1, a_2, a_3) = {} & (m+1-a_1)(m+1-a_2)(a_3+1) \\ & + (m+1-a_1)(m+1-a_3)(a_2+1) \\ & + (m+1-a_2)(m+1-a_3)(a_1+1) \\ & + (m+1-a_1)(m+1-a_2)(m+1-a_3), \end{aligned}$$

$$G_{m+1}(a_1, a_2, a_3) = m!(m-a_1)!(m-a_2)!(m-a_3)!/(m-a_1-a_2-a_3)!,$$

with $B(m+1) = \sum_{A_m} G_{m+1}(a_1, a_2, a_3)$ by Lemma 1.

Lemma 2. $B(m+2) = \sum_{A_m} F_{m+1}(a_1, a_2, a_3) G_{m+1}(a_1, a_2, a_3)$.

Proof. Given $(a_1, a_2, a_3) \in A_m$, $G_{m+1}(a_1, a_2, a_3)$ is the number of profiles on $m+1$ alternatives in which x beats each of the other m alternatives by a simple majority. When the $(m+2)$nd alternative is added to such a profile, it will be beaten by x under simple majority if and only if no more than one voter has the new alternative ranked ahead of x, and $F_{m+1}(a_1, a_2, a_3)$ is the number of ways this can be done. Therefore $B(m+2)$ equals $F_{m+1}(a_1, a_2, a_3) G_{m+1}(a_1, a_2, a_3)$ summed over A_m. □

Lemma 3. *If integers $a \geq 0$ and $b > 0$ satisfy $b \geq a$, then*

$$1+\frac{a}{b}+\frac{a(a-1)}{b(b-1)}+\cdots+\frac{a(a-1)\cdots 1}{b(b-1)\cdots(b-a+1)}=\frac{b+1}{b-a+1}.$$

Proof. The conclusion clearly holds if $a=0$ or $b=1$. For $a>0$ and $b>1$, the sum in Lemma 3 equals $1+(a/b)$ [sum for $a-1$ and $b-1$]. If the lemma's conclusion holds for $b-1$ and all $0 \leq a \leq b-1$, then the latter [sum] is $b/(b-a+1)$, with $1+(a/b)[b/(b-a+1)]=(b+1)/(b-a+1)$. The lemma then follows from induction on b. □

Lemma 4. *If integers $a_1 \geq 0$ and $a_2 \geq 0$ satisfy $a_1+a_2 \leq m$, then*

$$\sum_{a_3=0}^{m-a_1-a_2} F_{m+1}(a_1, a_2, a_3)G_{m+1}(a_1, a_2, a_3) \leq$$
$$\leq (m+2)^2(m+1) \sum_{a_3=0}^{m-a_1-a_2} G_{m+1}(a_1, a_2, a_3),$$

with $<$ in the conclusion if $a_1+a_2>0$.

Proof. Given $a_1 \geq 0$, $a_2 \geq 0$ and $a_1+a_2 \leq m$, the definition of G_{m+1} prior to Lemma 2 implies that

$$\sum_{a_3=0}^{m-a_1-a_2} G_{m+1}(a_1, a_2, a_3)=\frac{m!(m-a_1)!(m-a_2)!}{(m-a_1-a_2)!}(m!)$$
$$\times\left[1+\frac{m-a_1-a_2}{m}+\cdots+\frac{(m-a_1-a_2)\cdots 1}{m(m-1)\cdots(a_1+a_2+1)}\right]$$
$$=\frac{m!(m-a_1)!(m-a_2)!(m+1)!}{(m-a_1-a_2)!(a_1+a_2+1)} \quad \text{by Lemma 3.}$$

When the first and fourth terms in the definition of F_{m+1} that precedes Lemma 2 are grouped, we get

$$\sum_{a_3=0}^{m-a_1-a_2} F_{m+1}(a_1, a_2, a_3)G_{m+1}(a_1, a_2, a_3)=$$
$$=\sum_{a_3}(m+2)(m+1-a_1)(m+1-a_2)G_{m+1}(a_1, a_2, a_3)$$
$$+\sum_{a_3}[(m+1-a_1)(a_2+1)+(m+1-a_2)(a_1+1)]$$
$$\times(m+1-a_3)G_{m+1}(a_1, a_2, a_3)$$
$$=(m+2)(m+1-a_1)(m+1-a_2)\frac{m!(m-a_1)!(m-a_2)!(m+1)!}{(m-a_1-a_2)!(a_1+a_2+1)}$$
$$+[(m+1-a_1)(a_2+1)+(m+1-a_2)(a_1+1)]$$
$$\times\frac{m!(m-a_1)!(m-a_2)!(m+1)!(m+2)}{(m-a_1-a_2)!(a_1+a_2+2)}$$

by the result just proved and Lemma 3. The ratio of

$$\sum_{a_3} F_{m+1}(a_1, a_2, a_3)G_{m+1}(a_1, a_2, a_3)$$

to

$$\sum_{a_3} G_{m+1}(a_1, a_2, a_3)$$

is therefore

$$\begin{aligned}&(m+2)(m+1-a_1)(m+1-a_2)+(m+2)[(m+1-a_1)(a_2+1)\\&+(m+1-a_2)(a_1+1)](a_1+a_2+1)/(a_1+a_2+2)=\\&=\frac{m+2}{a_1+a_2+2}[(a_1+a_2+2)m^2+(3a_1+3a_2+6)m+f(a_1, a_2)]\end{aligned}$$

where

$$f(a_1, a_2)=4+a_1+a_2-a_1^2-a_2^2-a_1^2a_2-a_1a_2^2-2a_1a_2.$$

Since

$$(m+2)^2(m+1)=\frac{m+2}{a_1+a_2+2}\times[(a_1+a_2+2)m^2+(3a_1+3a_2+6)m+(2a_1+2a_2+4)],$$

and since $f(a_1, a_2)\leq 2a_1+2a_2+4$ with $<$ holding if $a_1+a_2>0$, it follows that

$$\sum_{a_3} F_{m+1}(a_1, a_2, a_3)G_{m+1}(a_1, a_2, a_3)\leq(m+2)^2(m+1)\sum_{a_3} G_{m+1}(a_1, a_2, a_3),$$

with $<$ holding if $a_1+a_2>0$. □

Finally, when both sides of the inequality in Lemma 4 are summed over a_1 and a_2 for $a_1\geq 0$, $a_2\geq 0$ and $a_1+a_2\leq m$, we get

$$\sum_{A_m} F_{m+1}(a_1, a_2, a_3)G_{m+1}(a_1, a_2, a_3)<(m+2)^2(m+1)\sum_{A_m} G_{m+1}(a_1, a_2, a_3),$$

given $m\geq 1$. Then (1) follows immediately from Lemmas 1 and 2.

3. Conjecture 2 for $m=3$ and odd n

For convenience in this and the next two sections we shall let $C(n)=C(3, n)$, so that $C(n)$ is the number of n-term Condorcet profiles on three alternatives divided by 6^n. Our purpose in the present section is to prove

Theorem 2. *$C(n) > C(n+2)$ for all odd $n \geqslant 1$.*

We shall prove this using the vernacular of the random-voters model with $\{1,2,3\}$ as the set of three alternatives. The main part of the proof will be preceded by the following lemma in which, for *even n*,

$Q_n(t)$ = Probability {(number of voters who prefer 1 to 3) − (number of voters who prefer 3 to 1) $= t \mid n$ is *even*, $\frac{1}{2}n$ voters prefers 1 to 2 and the other $\frac{1}{2}n$ voters prefer 2 to 1}.

By the symmetry of the conditioning event, $Q_n(t) = Q_n(-t)$ for $t \in \{2, 4, \ldots, n\}$.

Lemma 5. *If $n \geqslant 2$ is even, then $Q_n(0) > Q_n(2) > Q_n(4) > \cdots > Q_n(n)$.*

Proof. Since $Q_2(0) = \frac{5}{9}$ and $Q_2(2) = Q_2(-2) = \frac{2}{9}$, the conclusion holds for $n = 2$. Assume henceforth that n is even $n \geqslant 4$ and $t \in \{-n, -n+2, \ldots, n-2, n\}$. Although n is fixed in the following, the proof applies to any n as just specified. By dividing the n voters into a subset of two voters and a second subset of the other $n-2$ voters such that, within each subset the same number of voters prefer 1 to 2 as prefer 2 to 1, it is evident that

$$Q_n(t) = \tfrac{2}{9}Q_{n-2}(t-2) + \tfrac{5}{9}Q_{n-2}(t) + \tfrac{2}{9}Q_{n-2}(t+2).$$

Each Q_{n-2} term on the right-hand side can be decomposed in like manner in terms of Q_{n-4} and Q_2 (which provides the $\frac{2}{9}$ and $\frac{5}{9}$ multipliers). In general, for each positive integer k for which $n-2k \geqslant 2$, there are integers $f_k(a)$ for $a \in \{0, 2, 4, \ldots, 2k\}$ such that

$$9^k Q_n(t) = f_k(0)Q_{n-2k}(t) + \sum_{j=1}^{k} f_k(2j)[Q_{n-2k}(t-2j) + Q_{n-2k}(t+2j)]$$

with

$$f_k(0) > f_k(2) > \cdots > f_k(2k) > 0.$$

The previous expression for $Q_n(t)$ in terms of Q_{n-2} shows that this holds for $k=1$ with $f_1(0) = 5$ and $f_1(2) = 2$. This completes the proof for $n = 4$. When $n \geqslant 6$, induction on k then shows that it is generally true: assuming its truth for all k up to some k for which $n - 2k \geqslant 4$, the breakdown of each Q_{n-2k} in terms of $Q_{n-2k-2} = Q_{n-2(k+1)}$ shows that

$$f_{k+1}(a) = 2f_k(a-2) + 5f_k(a) + 2f_k(a+2)$$

for each $a \in \{0, 2, \ldots, 2(k+1)\}$ where, by convention, $f_k(a)=0$ for $a>2k$ and $f_k(-2)=f_k(2)$; then for $a \in \{0, 2, \ldots, 2(k+1)\}$, $f_{k+1}(a)>f_{k+1}(a+2)$ iff $2f_k(a-2)+3f_k(a)>3f_k(a+2)+2f_k(a+4)$, which is true by the induction hypothesis.

Since $Q_2(a)\neq 0$ iff $a \in \{-2, 0, 2\}$, the conclusion just proved shows that, when $k=\frac{1}{2}(n-2)$ and $t \in \{0, 2, \ldots, n\}$,

$$9^k Q_n(t)=f_k(t-2)Q_2(2)+f_k(t)Q_2(0)+f_k(t+2)Q_2(-2)$$
$$=\tfrac{2}{9}f_k(t-2)+\tfrac{5}{9}f_k(t)+\tfrac{2}{9}f_k(t+2),$$

with $f_k(n+2)=0$ and $f_k(-2)=f_k(2)$. Hence, when $k=\frac{1}{2}(n-2)$ and $t \in \{0, 2, \ldots, n-2\}$, $Q_n(t)>Q_n(t+2)$ iff $2f_k(t-2)+3f_k(t)>3f_k(t+2)+2f_k(t+4)$, which is true since $f_k(0)>f_k(2)>\cdots>f_k(n-2)>0$. □

We now prove Theorem 2. Let $P_1(n)$ be the probability that alternative 1 is the Condorcet alternative when there are $n \geq 1$ voters and n is odd. Since n is odd, symmetry implies that $C(n)=3P_1(n)$. Hence to prove Theorem 2 we shall show that

$$P_1(n)>P_1(n+2) \quad \text{for all odd } n \geq 1.$$

Since $P_1(1)=\frac{1}{3}>P_1(3)=\frac{17}{54}$, this is true for $n=1$. We assume henceforth that $n \geq 3$.

Given n voters with n odd, let (α, β) denote the event in which (i) α of the alternatives in $\{2, 3\}$ are beaten by alternative 1 by exactly one vote (e.g., $\frac{1}{2}(n+1)$ voters prefer 1 to 2 and the other $\frac{1}{2}(n-1)$ perfer 2 to 1), (ii) β alternatives in $\{2, 3\}$ beat alternative 1 by exactly one vote, and (iii) the other $2-(\alpha+\beta)$ alternatives in $\{2, 3\}$ are beaten by 1 by three or more votes. Let $P(\alpha, \beta)$ be the probability that (α, β) obtains. Then

$$P_1(n)=P(0,0)+P(1,0)+P(2,0).$$

In addition, when two new voters are added to the n voters who are represented in (α, β), we see that

$$P_1(n+2)=P(0,0)+\tfrac{3}{4}P(1,0)+\tfrac{11}{18}P(2,0)$$
$$+\tfrac{1}{4}P(0,1)+\tfrac{1}{9}P(0,2)+\tfrac{2}{9}P(1,1).$$

For example, if $(\alpha, \beta)=(2,0)$ obtains for the first n voters, then alternative 1 will be the Condorcet alternative for the $n+2$ voters iff at least one of the two new voters prefers 1 to 2 *and* at least one of the two new voters prefers 1 to 3. The probability of the latter joint event for the two new voters is $\frac{11}{18}$.

According to the preceding paragraph, $P_1(n) > P_1(n+2)$ iff

$$9P(1,0)+14P(2,0)>9P(0,1)+4P(0,2)+8P(1,1).$$

To verify the latter inequality we use a conditional analysis of the $P(\alpha, \beta)$ for the n original voters as follows. Let A and A' be respectively the event that 1 beats 2 by one vote (for the n voters) and the event that 2 beats 1 by one vote. By symmetry, $P(A)=P(A')$, where $P(\ )$ denotes the probability of the event in parentheses. Conditional analysis and symmetry considerations yield

$$\begin{aligned}
P(1,0) &= 2P(1 \text{ beats } 3 \text{ by } \geq 3 \text{ votes} \mid A)P(A),\\
P(2,0) &= P(1 \text{ beats } 3 \text{ by } 1 \text{ vote} \mid A)P(A),\\
P(0,1) &= 2P(1 \text{ beats } 3 \text{ by } \geq 3 \text{ votes} \mid A')P(A)\\
&= 2P(2 \text{ beats } 3 \text{ by } \geq 3 \text{ votes} \mid A)P(A),\\
P(0,2) &= P(2,0),\\
P(1,1) &= 2P(3 \text{ beats } 1 \text{ by } 1 \text{ vote} \mid A)P(A).
\end{aligned}$$

When these are substituted into the preceding inequality and $P(A)>0$ is canceled, we see that $P_1(n) > P_1(n+2)$ if

$$\begin{aligned}
&9P(1 \text{ beats } 3 \text{ by } \geq 3 \text{ votes} \mid A)+5P(1 \text{ beats } 3 \text{ by } 1 \text{ vote} \mid A) >\\
&\quad > 9P(2 \text{ beats } 3 \text{ by } \geq 3 \text{ votes} \mid A)+8P(3 \text{ beats } 1 \text{ by } 1 \text{ vote} \mid A).
\end{aligned}$$

Since these conditional probabilities are not affected by the specific manner in which A is realized – i.e., which voters prefer 1 to 2 and 2 to 1 – we fix one voter who prefers 1 to 2 and let the remaining $n-1$ voters be evenly divided between 1 versus 2. Then, with $Q_{n-1}(t)$ referring to the latter $n-1$ voters, it follows from the definition of $Q_{n-1}(t)$ that

$$\begin{aligned}
&P(1 \text{ beats } 3 \text{ by } \geq 3 \text{ votes} \mid A) = \tfrac{2}{3}Q_{n-1}(2) + \sum_{t \geq 4} Q_{n-1}(t),\\
&P(1 \text{ beats } 3 \text{ by } 1 \text{ vote} \mid A) = \tfrac{2}{3}Q_{n-1}(0) + \tfrac{1}{3}Q_{n-1}(2),\\
&P(2 \text{ beats } 3 \text{ by } \geq 3 \text{ votes} \mid A) = \tfrac{1}{3}Q_{n-1}(2) + \sum_{t \geq 4} Q_{n-1}(t),\\
&P(3 \text{ beats } 1 \text{ by } 1 \text{ vote} \mid A) = \tfrac{2}{3}Q_{n-1}(2) + \tfrac{1}{3}Q_{n-1}(0).
\end{aligned}$$

The final equation here uses the fact that $Q_{n-1}(-2) = Q_{n-1}(2)$, and the third equation uses symmetry as follows: (2 beats 3 by ≥ 3 votes $\mid A$) iff {2 beats 3 by 2 votes in the $n-1$ (whose probability, by symmetry, is the same as 1 beating 3 by 2 votes, i.e., $Q_{n-1}(2)$) and the fixed voter prefers 2 to 3 (with probability $\tfrac{1}{3}$)} or {2 beats 3 by 4 or more votes (with probability $Q_{n-1}(4)+Q_{n-1}(6)+\cdots$ by symmetry)}. When the four preceding equalities are substituted into the former inequality for $P_1(n)>P_1(n+2)$, we get $P_1(n)>P_1(n+2)$ iff $Q_{n-1}(0)>Q_{n-1}(2)$, which is true by Lemma 5 for odd $n\geqslant 3$.

4. Some results for $m=3$ and even n

The obvious fact that differentiates even n from odd n is that a Condorcet profile for even n can have more than one Condorcet alternative. Moreover, even when an even-n profile has a unique Condorcet alternative, another alternative can tie the Condorcet alternative so long as it is beaten by something else. For example, we can have 1 and 2 tied with 1 beating 3 and 3 beating 2. Then 1 is the unique Condorcet alternative although it does not have strict majorities over both of the other alternatives.

This section will focus on a specific alternative either as the unique Condorcet alternative that beats the others or as one of the possibly more than one Condorcet alternatives. Kelly's Conjecture 2 for three alternatives and even n will be examined in the next section. According to our present concerns, let {1, 2, 3} be the set of three alternatives and let

$$P_1(n)=\text{Probability (alternative 1 is a Condorcet alternative and neither 2 nor 3 ties } 1 \mid n \text{ voters)},$$

$$T_1(n)=\text{Probability (alternative 1 is a Condorcet alternative} \mid n \text{ voters)}.$$

If n is odd, then $T_1(n)=P_1(n)$, but if n is even, then $T_1(n)>P_1(n)$ for $n\geqslant 2$.

Theorem 3. *$P_1(n)<P_1(n+2)$ for all even $n\geqslant 2$.*

Theorem 4. *$T_1(n)>T_1(n+2)$ for all even $n\geqslant 2$.*

Theorems 2 through 4 show that T_1 for even n behaves in the same way as P_1 for odd n, while P_1 for even n reverses the behavior of P_1 for odd n. Since ties for the even-n case have probability zero in the limit of n, the limit probability $P_1(\infty)$—which is one-third of Guilbaud's number [8, 9] and is approximately 0.30408— is approached from above by P_1 for odd n and T_1 for even n but is approached from below by P_1 for even n.

Before proving Theorems 3 and 4 we shall establish a lemma that will be used in the proof of Theorem 3.

Lemma 6. *$n[Q_{n-2}(0)-Q_{n-2}(4)]=9Q_n(2)$ for all even $n\geqslant 4$.*

Proof. Let $2r=n-2$ and partition the $2r$ voters who are evenly divided on 1 versus 2 in the conditioning event for $Q_{n-2}(t)$ into r groups of two voters each such that, within each group, one voter prefers 1 to 2 and the other prefers 2 to 1. Then

$$n[Q_{n-2}(0)-Q_{n-2}(4)]/2=(r+1)[Q_{2r}(0)-Q_{2r}(4)]$$
$$=(r+1)\left\{\sum_{k=0}^{[r/2]}\frac{r!2^{2k}5^{r-2k}}{(k!)^2(r-2k)!9^r}-\sum_{k=2}^{[1+r/2]}\frac{r!2^{2k-2}5^{r-2k+2}}{k!(k-2)!(r-2k+2)!9^r}\right\},$$

which follows from the within-group probabilities of $\frac{2}{9}$, $\frac{2}{9}$ and $\frac{5}{9}$ respectively that both voters prefer 1 to 3, both prefer 3 to 1, and the two are divided on 1 vs. 3. When the $k=0$ term is separated from the first sum and k is replaced by $k+1$ throughout the second sum, the preceding expression can be written as

$$(r+1)\left\{(\tfrac{5}{9})^r+\sum_{k=1}^{[r/2]}\frac{r!2^{2k}5^{r-2k}}{(k!)^2(r-2k)!9^r}-\sum_{k=1}^{[r/2]}\frac{r!2^{2k}5^{r-2k}}{(k+1)!(k-1)!(r-2k)!9^r}\right\}$$
$$=(r+1)\left\{(\tfrac{5}{9})^r+\sum_{k=1}^{[r/2]}\frac{r!2^{2k}5^{r-2k}}{k!(k+1)!(r-2k)!9^r}\right\}$$
$$=\tfrac{9}{2}\sum_{k=0}^{[r/2]}\frac{(r+1)!2^{2k+1}5^{r-2k}}{k!(k+1)!(r-2k)!9^{r+1}}=\tfrac{9}{2}Q_{2(r+1)}(2).$$

Therefore $n[Q_{n-2}(0)-Q_{n-2}(4)]=9Q_n(2)$. □

Proof of Theorem 3. This proof is similar to the proof of Theorem 2 but is slightly more complex for reasons that will become apparent. Throughout the proof, n is even and $n\geqslant 2$. In this context we let (α,β) denote the event in which (i) α alternatives in $\{2,3\}$ are beaten by alternative 1 by exactly *two* votes, (ii) β alternatives in $\{2,3\}$ tie alternative 1 (e.g., $\frac{1}{2}n$ voters prefer 1 to 2 and the other $\frac{1}{2}n$ prefer 2 to 1), and (iii) the other $2-(\alpha+\beta)$ alternatives in $\{2,3\}$ are beaten by alternative 1 by four or more votes. With $P(\alpha,\beta)$ the probability of event (α,β) for n voters, the addition of two new voters to the n voters represented in (α,β) gives

$$P_1(n+2)-P_1(n)=[P(0,0)+\tfrac{3}{4}P(1,0)+\tfrac{11}{18}P(2,0)$$
$$+\tfrac{1}{4}P(0,1)+\tfrac{1}{9}P(0,2)+\tfrac{2}{9}P(1,1)]$$
$$-[P(0,0)+P(1,0)+P(2,0)].$$

It follows from this and symmetry considerations that $P_1(n+2)>P_1(n)$ iff

$$9p_1+8p_2+2p_3>9p_4+7p_5$$

where, given n voters,

$$p_1 = P(1 \text{ ties } 2,\ 1 \text{ beats } 3 \text{ by } \geq 4 \text{ votes}),$$
$$p_2 = P(1 \text{ ties } 2,\ 1 \text{ beats } 3 \text{ by } 2 \text{ votes}),$$
$$p_3 = P(1 \text{ ties } 2,\ 1 \text{ ties } 3),$$
$$p_4 = P(1 \text{ beats } 2 \text{ by } 2 \text{ votes},\ 1 \text{ beats } 3 \text{ by } \geq 4 \text{ votes}),$$
$$p_5 = P(1 \text{ beats } 2 \text{ by } 2 \text{ votes},\ 1 \text{ beats } 3 \text{ by } 2 \text{ votes}).$$

Let A and B be respectively the event that 1 ties 2 and the event that 1 beats 2 by 2 votes. Then

$$p_1 = P(1 \text{ beats } 3 \text{ by } \geq 4 \text{ votes} \mid A)P(A) = p_1' P(A),$$
$$p_2 = P(1 \text{ beats } 3 \text{ by } 2 \text{ votes} \mid A)P(A) = p_2' P(A),$$
$$p_3 = P(1 \text{ ties } 3 \mid A)P(A) = p_3' P(A),$$
$$p_4 = P(1 \text{ beats } 3 \text{ by } \geq 4 \text{ votes} \mid B)P(B) = p_4' P(B),$$
$$p_5 = P(1 \text{ beats } 3 \text{ by } 2 \text{ votes} \mid B)P(B) = p_5' P(B),$$

where p_1' through p_5' are defined in context. Since $P(A) = 2^{-n}\binom{n}{n/2}$ and $P(B) = 2^{-n}\binom{n}{n/2+1}$, $P(A)/P(B) = 1 + 2/n$. This fact and substitution of p_1' through p_5' into the preceding inequality for $P_1(n+2) > P_1(n)$ yields $P_1(n+2) > P_1(n)$ iff

$$(1+2/n)(9p_1' + 8p_2' + 2p_3') > 9p_4' + 7p_5'.$$

It is easily checked that this holds for $n = 2$, so we assume henceforth that $n \geq 4$. By viewing A as composed of two voters who are divided on 1 versus 2 plus the remaining $n-2$ voters who are evenly split between 1 preferred to 2 and 2 preferred to 1, it follows that

$$p_1' = \tfrac{2}{9}Q_{n-2}(2) + \tfrac{7}{9}Q_{n-2}(4) + Q_{n-2}(\geq 6),$$
$$p_2' = \tfrac{2}{9}Q_{n-2}(0) + \tfrac{5}{9}Q_{n-2}(2) + \tfrac{2}{9}Q_{n-2}(4),$$
$$p_3' = \tfrac{4}{9}Q_{n-2}(2) + \tfrac{5}{9}Q_{n-2}(0),$$

where $Q_{n-2}(\geq 6) = Q_{n-2}(6) + Q_{n-2}(8) + \cdots$, and where $Q_{n-2}(-2) = Q_{n-2}(2)$ has been used in p_3'. Similarly, by viewing B as composed of two voters who both prefer 1 to 2 plus $n-2$ voters who are evenly split on 1 versus 2,

$$p_4' = \tfrac{4}{9}Q_{n-2}(2) + \tfrac{8}{9}Q_{n-2}(4) + Q_{n-2}(\geq 6),$$
$$p_5' = \tfrac{4}{9}Q_{n-2}(0) + \tfrac{4}{9}Q_{n-2}(2) + \tfrac{1}{9}Q_{n-2}(4).$$

Substitution into the preceding inequality gives $P_1(n+2) > P_1(n)$ iff

$$(1+2/n)[26Q_{n-2}(0) + 66Q_{n-2}(2) + 79Q_{n-2}(4) + 81Q_{n-2}(\geq 6)] >$$
$$> [28Q_{n-2}(0) + 64Q_{n-2}(2) + 79Q_{n-2}(4) + 81Q_{n-2}(\geq 6)].$$

Since the sum of the $Q_{n-2}(t)$ for even t equals 1, and since $Q_{n-2}(-t)=Q_{n-2}(t)$ by the line before Lemma 5, we have

$$Q_{n-2}(\geq 6)=\tfrac{1}{2}-\tfrac{1}{2}Q_{n-2}(0)-Q_{n-2}(2)-Q_{n-2}(4).$$

When this is substituted into the preceding inequality and fractions are cleared, we get $P_1(n+2)>P_1(n)$ iff

$$81>2n[Q_{n-2}(0)-Q_{n-2}(2)]+29Q_{n-2}(0)+30Q_{n-2}(2)+4Q_{n-2}(4).$$

Since Lemmas 5 and 6 imply that $9Q_n(2)>n[Q_{n-2}(0)-Q_{n-2}(2)]$, it follows that the right-hand side of the preceding inequality cannot exceed $2(9)+30=48$. □

Proof of Theorem 4. In the T_1 context let (α,β) be the event for n voters in which (i) α alternatives in $\{2,3\}$ tie alternative 1, (ii) β alternatives in $\{2,3\}$ beat alternative 1 by exactly two votes, and (iii) the other $2-(\alpha+\beta)$ alternatives in $\{2,3\}$ are beaten by alternative 1 by two or more votes. Then, proceeding as in previous proofs with $P(\alpha,\beta)$ the probability of (α,β),

$$T_1(n)-T_1(n+2)=[P(0,0)+P(1,0)+P(2,0)]-[P(0,0)+\tfrac{3}{4}P(1,0)+\tfrac{11}{18}P(2,0)+\tfrac{1}{4}P(0,1)+\tfrac{1}{9}P(0,2)+\tfrac{2}{9}P(1,1)].$$

Let A and B be respectively the event that 1 ties 2 and the event that 2 beats 1 by 2 votes, with respect to n voters. Then

$$P(1,0)=2P(1 \text{ beats } 3 \text{ by } \geq 2 \text{ votes} \mid A)P(A)=2q_1P(A),$$

$$P(2,0)=P(1 \text{ ties } 3 \mid A)P(A)=q_2P(A),$$

$$P(0,1)=2P(1 \text{ beats } 3 \text{ by } \geq 2 \text{ votes} \mid B)P(B)=2q_3P(B),$$

$$P(1,1)=2P(1 \text{ ties } 3 \mid B)P(B)=2q_4P(B),$$

$$P(0,2)=P(3 \text{ beats } 1 \text{ by } 2 \text{ votes} \mid B)P(B)=q_5P(B),$$

where q_1 through q_5 are defined in context. Since $P(A)=\binom{n}{n/2}2^{-n}$ and $P(B)=P(A)n/(n+2)$, it follows that

$$T_1(n)-T_1(n+2)=\frac{\binom{n}{n/2}2^{-n}}{18(n+2)}[(9q_1+7q_2)(n+2)-(9q_3+8q_4+2q_5)n].$$

Since this is easily seen to be positive for $n=2$, $n\geq 4$ will be presumed henceforth. Then, since reasoning that follows that used to break down p_1' through p_5' in the preceding proof shows that

$$q_1 = \tfrac{2}{9}Q_{n-2}(0) + \tfrac{7}{9}Q_{n-2}(2) + Q_{n-2}(\geq 4),$$
$$q_2 = \tfrac{2}{9}Q_{n-2}(-2) + \tfrac{5}{9}Q_{n-2}(0) + \tfrac{2}{9}Q_{n-2}(2),$$
$$q_3 = \tfrac{1}{9}Q_{n-2}(0) + \tfrac{5}{9}Q_{n-2}(2) + Q_{n-2}(\geq 4),$$
$$q_4 = \tfrac{1}{9}Q_{n-2}(-2) + \tfrac{4}{9}Q_{n-2}(0) + \tfrac{4}{9}Q_{n-2}(2),$$
$$q_5 = \tfrac{1}{9}Q_{n-2}(-4) + \tfrac{4}{9}Q_{n-2}(-2) + \tfrac{4}{9}Q_{n-2}(0),$$

and since $Q_{n-2}(-t) = Q_{n-2}(t)$, it follows that

$$T_1(n) - T_1(n+2) = \frac{\binom{n}{n/2}2^{-n}}{81(n+2)}\{n[2Q_{n-2}(0) - Q_{n-2}(2) - Q_{n-2}(4)] + [53Q_{n-2}(0) + 91Q_{n-2}(2) + 81Q_{n-2}(\geq 4)]\}.$$

Since Lemma 5 implies that the first term in brackets is positive, and since the second term in brackets is positive too, it follows that $T_1(n) > T_1(n+2)$ for all even n. □

Note on T_1. The preceding expression for $T_1(n) - T_1(n+2)$ implies a limit result that will be used in the next section. Here and later we shall let $f(n) \sim g(n)$ mean that $|f(n) - g(n)| \to 0$ as $n \to \infty$. In addition, we define K_n by

$$K_n = \frac{\binom{n}{n/2}2^{-n}}{(n+2)}.$$

Lemma 7. $3[T_1(n) - T_1(n+2)]/K_n \sim \frac{3}{2}$.

Proof. When $Q_{n-2}(\geq 4) = \frac{1}{2} - \frac{1}{2}Q_{n-2}(0) - Q_{n-2}(2)$ is used in the final equation of the proof of Theorem 4, we get

$$3[T_1(n) - T_1(n+2)]/K_n = \tfrac{3}{2} + \tfrac{1}{27}n[2Q_{n-2}(0) - Q_{n-2}(2) - Q_{n-2}(4)] + \tfrac{1}{27}[12.5Q_{n-2}(0) + 10Q_{n-2}(2)].$$

In view of Lemmas 5 and 6, all terms on the right hand side go to zero as n gets large except for $\frac{3}{2}$. □

5. Conjecture 2 for $m = 3$ and even n

In this section we shall prove that Kelly's second conjecture holds for almost all even $n \geq 4$ when there are three alternatives. Throughout, the set of alternatives is $\{1, 2, 3\}$ with $C(n) = C(3, n)$.

Theorem 5. *$C(n) > C(n+2)$ for all even n greater than some positive integer N.*

In addition, $C(n) > C(n+2)$ can be established for small even $n \geq 4$ by direct calculation. For example, $C(4) = 1$ and $C(6) = 1291/1296$. Although we do not presently have a proof of $C(n) > C(n+2)$ for all even $n \geq 4$, we have no reason to doubt its validity.

Along with $T_1(n)$ from the preceding section let

$$T_{12}(n) = \text{Probability (alternatives 1 and 2 are Condorcet alternatives} \mid n \text{ voters)},$$

$$T_{123}(n) = \text{Probability (alternatives 1, 2 and 3 are Condorcet alternatives} \mid n \text{ voters)}.$$

Since symmetry among alternatives implies that

$$C(n) = 3T_1(n) - 3T_{12}(n) + T_{123}(n),$$

$$C(n) - C(n+2) = 3[T_1(n) - T_1(n+2)] - \{3[T_{12}(n) - T_{12}(n+2)] + [T_{123}(n+2) - T_{123}(n)]\}.$$

The limiting behavior of $3[T_1(n) - T_1(n+2)]$ is known from Lemma 7. We shall therefore focus on the term in braces in the preceding expression. Let

$Q_n(t_1, t_2) =$ Probability {(number of voters who prefer 1 to 3) − (number who prefer 3 to 1) $= t_1$, (number of voters who prefer 2 to 3) − (number who prefer 3 to 2) $= t_2 \mid n$ voters, $\frac{1}{2}n$ of whom prefer 1 to 2 and $\frac{1}{2}n$ of whom prefer 2 to 1}.

In all cases, t_1 and t_2 are even integers and n is a positive even integer. By symmetry, $Q_n(t_1, t_2) = Q_n(t_2, t_1) = Q_n(-t_1, -t_2)$. For notational convenience we shall let $Q_n(t_1, \geq t_2) = Q_n(t_1, t_2) + Q_n(t_1, t_2+2) + \cdots$ with similar conventions for $Q_n(\geq t_1, t_2)$ and $Q_n(\geq t_1, \geq t_2)$.

To analyze the part of $C(n) - C(n+2)$ that involves T_{12} and T_{123} we observe first that

$$T_{12}(n) = \binom{n}{n/2} 2^{-n} Q_n(\geq 0, \geq 0),$$

$$T_{123}(n) = \binom{n}{n/2} 2^{-n} Q_n(0, 0),$$

$$T_{12}(n+2) = \binom{n+2}{1+n/2} 2^{-n-2} Q_{n+2}(\geq 0, \geq 0)$$

$$= \binom{n}{n/2} 2^{-n} \left(\frac{n+1}{n+2}\right) Q_{n+2}(\geq 0, \geq 0),$$

$$T_{123}(n+2) = \binom{n}{n/2} 2^{-n} \left(\frac{n+1}{n+2}\right) Q_{n+2}(0, 0).$$

A straightforward conditional analysis on two voters who are evenly divided on 1 versus 2 shows that

$$\begin{aligned}Q_{n+2}(\geq 0, \geq 0) = \tfrac{1}{9}[&9Q_n(\geq 2, \geq 2) + 7Q_n(\geq 2, 0) + 7Q_n(0, \geq 2)\\ &+ 6Q_n(0,0) + 2Q_n(\geq 2, -2) + 2Q_n(-2, \geq 2)\\ &+ 2Q_n(0,-2) + 2Q_n(-2,0) + Q_n(-2,-2)]\\ = \tfrac{1}{9}[&9Q_n(\geq 2, \geq 2) + Q_n(2,2) + 14Q_n(0, \geq 2)\\ &+ 6Q_n(0,0) + 4Q_n(-2, \geq 0)].\end{aligned}$$

In like manner,

$$Q_{n+2}(0,0) = \tfrac{1}{9}[2Q_n(2,2) + 4Q_n(0,2) + 3Q_n(0,0)].$$

Using

$$Q_n(\geq 2, \geq 2) = Q_n(\geq 0, \geq 0) - 2Q_n(0, \geq 0) + Q_n(0,0)$$

and $Q_n(0, \geq 2) = Q_n(0, \geq 0) - Q_n(0,0)$, it follows that

$$\begin{aligned}&\{3[T_{12}(n) - T_{12}(n+2)] + [T_{123}(n+2) - T_{123}(n)]\}/K_n =\\ &\quad = 3Q_n(\geq 0, \geq 0) - \tfrac{1}{9}[(9n+18)Q_n(0,0) - (4n+4)Q_n(0,2)\\ &\qquad + (n+1)Q_n(2,2)]\\ &\qquad + \tfrac{4}{3}(n+1)[Q_n(0, \geq 0) - Q_n(-2, \geq 0)].\end{aligned}$$

The limiting behaviors of the three parts of this expression are covered by the following three lemmas. As before, $f(n) \sim g(n)$ means that $|f(n) - g(n)| \to 0$ as $n \to \infty$. In addition, we shall write $f(n) \lesssim g(n)$ if and only if either $f(n) < g(n)$ for all large even n or else $f(n) \sim g(n)$, and $f(n) \gtrsim g(n)$ if and only if $g(n) \lesssim f(n)$.

Lemma 8. $Q_n(\geq 0, \geq 0) \sim \frac{1}{3}$.

Lemma 9. $nQ_n(t_1, t_2) \sim 3\sqrt{3}/(2\pi)$ for fixed (t_1, t_2).

Lemma 10. $(n+1)[Q_n(0, \geq 0) - Q_n(-2, \geq 0)] \lesssim 9/(4\pi)$.

The proofs of these lemmas will be deferred until we see how they are used in proving Theorem 5.

Proof of Theorem 5. Lemmas 7 through 10 along with $Q_n(t_1, t_2) \sim 0$ show that, since

$$[C(n)-C(n+2)]/K_n = 3[T_1(n)-T_1(n+2)]/K_n - 3Q_n(\geq 0, \geq 0) + \tfrac{1}{9}[(9n+18)Q_n(0,0) - (4n+4)Q_n(0,2)+(n+1)Q_n(2,2)] - \tfrac{4}{3}(n+1)[Q_n(0,\geq 0)-Q_n(-2,\geq 0)],$$

$$[C(n)-C(n+2)]/K_n \geq \tfrac{3}{2}-1+\tfrac{1}{9}[6(3\sqrt{3})/(2\pi)]-\tfrac{4}{3}[9/(4\pi)] = \frac{1}{2}-\frac{3-\sqrt{3}}{\pi}.$$

Since $\frac{1}{2}-(3-\sqrt{3})/\pi$ is positive (about 0.1), $C(n)>C(n+2)$ for all large n. □

To approach the proofs of Lemmas 8 through 10 let the $n=2r$ voters in the conditioning event in the definition of $Q_n(t_1, t_2)$ be partitioned into r two-voter groups such that, within each group, one voter prefers 1 to 2 and the other prefers 2 to 1. For the ith two-voter group let x_i equal 1, 0, or -1 respectively according to whether both voters prefer 1 to 3, one prefers 1 to 3 and the other prefers 3 to 1, or both prefer 3 to 1; let y_i equal 1, 0, or -1 respectively according to whether both voters prefer 2 to 3, one prefers 2 to 3 and the other prefers 3 to 2, or both prefer 3 to 2. Seven of the nine (x_i, y_i) pairs are possible—only $(1,-1)$ and $(-1,1)$ are excluded. The probability for each pair is shown in the following matrix, followed in parentheses by the number of groups that have that pair.

		y: 1	y: 0	y: −1
x	1	$\frac{1}{9}(a_1)$	$\frac{1}{9}(a_2)$	0
	0	$\frac{1}{9}(a_3)$	$\frac{3}{9}(a_7)$	$\frac{1}{9}(a_4)$
	−1	0	$\frac{1}{9}(a_5)$	$\frac{1}{9}(a_6)$

For example, a_1 groups have $(x_i, y_i)=(1,1), \ldots,$ and a_7 groups have $(x_i, y_i)=(0,0)$, with $a_1+\cdots+a_7=r$. Note also that $t_1=2[a_1+a_2-a_5-a_6]$ and $t_2=2[a_1+a_3-a_4-a_6]$.

Proof of Lemma 8. The analysis of the preceding paragraph implies that each random vector (x_i, y_i) has mean $(0,0)$, covariance matrix

$$V=\begin{bmatrix} \frac{4}{9} & \frac{2}{9} \\ — & \frac{4}{9} \end{bmatrix}$$

and correlation coefficient $\rho=\frac{1}{2}$ between the two variables. Since $(\sum x_i/\sqrt{r}, \sum y_i/\sqrt{r})$ has a bivariate normal distribution with mean $(0,0)$ and $\rho=\frac{1}{2}$ as $n\to\infty$, standard tables (e.g., Yamauti [13]) show that its nonnegative orthant probability approaches $\frac{1}{3}$ as $n \to \infty$. Since $t_1 \geq 0 \leftrightarrow \sum x_i \geq 0$, and $t_2 \geq 0 \leftrightarrow \sum y_i \geq 0$, it follows that $Q_n(\geq 0, \geq 0) \sim \frac{1}{3}$. □

Proof of Lemma 9. The lattice distribution treatment for the central limit theorem in Bhattacharya and Ranga Rao [2] or Bhattacharya [1, Theorem 2.1] implies that, for fixed (t_1, t_2),

$$rQ_{2r}(t_1, t_2) \sim \phi_V(0, 0),$$

where ϕ_V is the bivariate normal density with mean $(0, 0)$ and covariance matrix V as given in the preceding proof. Since

$$\phi_V(0, 0) = [\text{determinant}\,(V^{-1})]^{1/2}/(2\pi) = \sqrt{27/4}(2\pi),$$

it follows that $nQ_n(t_1, t_2) \sim 3\sqrt{3}/(2\pi)$. □

Proof of Lemma 10. Lemmas 5 and 6 show that $r[Q_{2r}(0) - Q_{2r}(-2)] \sim 0$. Since $Q_{2r}(t) = Q_{2r}(t, \geq 0) + Q_{2r}(t, <0)$, and since $r[Q_{2r}(2, 0) - Q_{2r}(0, 0)] \sim 0$ according to Lemma 9,

$$\begin{aligned}
0 &\sim r[Q_{2r}(0) - Q_{2r}(-2)] \\
&= r\{[Q_{2r}(0, \geq 0) - Q_{2r}(-2, \geq 0)] - [Q_{2r}(-2, <0) - Q_{2r}(0, <0)]\} \\
&\sim r\{[Q_{2r}(0, \geq 0) - Q_{2r}(-2, \geq 0)] - [Q_{2r}(2, >0) - Q_{2r}(0, >0)] \\
&\qquad\qquad - [Q_{2r}(2, 0) - Q_{2r}(0, 0)]\} \\
&= r[Q_{2r}(0, \geq 0) - Q_{2r}(-2, \geq 0)] - r[Q_{2r}(2, \geq 0) - Q_{2r}(0, \geq 0)].
\end{aligned}$$

Therefore

$$r[Q_{2r}(0, \geq 0) - Q_{2r}(-2, \geq 0)] \sim r[Q_{2r}(2, \geq 0) - Q_{2r}(0, \geq 0)].$$

Then, since $Q_n(t_1, \geq 0) - Q_n(t_1', \geq 0) \sim 0$ for fixed t_1 and t_1',

$$\begin{aligned}
&(n+1)[Q_n(0, \geq 0) - Q_n(-2, \geq 0)] \\
&\quad \sim n[Q_n(0, \geq 0) - Q_n(-2, \geq 0)] \\
&\quad = 2r[Q_{2r}(0, \geq 0) - Q_{2r}(-2, \geq 0)] \\
&\quad \sim r[Q_{2r}(0, \geq 0) - Q_{2r}(-2, \geq 0)] + r[Q_{2r}(2, \geq 0) - Q_{2r}(0, \geq 0)] \\
&\quad = r[Q_{2r}(2, \geq 0) - Q_{2r}(-2, \geq 0)] \\
&\quad \sim (r+1)[Q_{2r}(2, \geq 0) - Q_{2r}(-2, \geq 0)].
\end{aligned}$$

Therefore

$$(n+1)[Q_n(0, \geq 0) - Q_n(-2, \geq 0)] \sim (r+1)[Q_{2r}(2, \geq 0) - Q_{2r}(-2, \geq 0)].$$

We shall now show that $(r+1)[Q_{2r}(2,\geq 0)-Q_{2r}(-2,\geq 0)]\leq 9/(4\pi)$, which establishes Lemma 10.

With the a_i as defined in the matrix prior to the proof of Lemma 8,

$$Q_{2r}(2,\geq 0)=\sum_{A_1}\frac{r!}{\prod(a_i!)}\left(\frac{3^{a_7}}{9^r}\right),$$

$$Q_{2r}(-2,\geq 0)=\sum_{A_2}\frac{r!}{\prod(a_i!)}\left(\frac{3^{a_7}}{9^r}\right),$$

where $A_1=\{\text{nonnegative integral }(a_1,\ldots,a_7)\colon a_1+a_2=a_5+a_6+1,\ a_1+a_3\geq a_4+a_6,\ \sum a_i=r\}$ and $A_2=\{\text{nonnegative integral }(a_1,\ldots,a_7)\colon a_1+a_2=a_5+a_6-1,\ a_1+a_3\geq a_4+a_6,\ \sum a_i=r\}$. Let

$$A_3=\Big\{(a_1,\ldots,a_7)\colon a_1+a_2=a_5+a_6,\ a_2>0,\ a_5>0,\ a_1+a_3\geq a_4+a_6,\ a_3+a_6<a_1+a_4,\ \sum a_i=r+1\Big\},$$

$$A_4=\{(a_2,a_6,k,b)\colon 2(a_2+a_6+k)+b=r+1\},$$

$$A_5=\{b\colon b\leq r+1,\ r+1-b\text{ is even}\},$$

where all variables are nonnegative integers. We shall prove that

$$\begin{aligned}
&(r+1)[Q_{2r}(2,\geq 0)-Q_{2r}(-2,\geq 0)]\\
&\sim\sum_{A_2\cup\{a_5>0\}}\frac{(r+1)![a_5-(a_2+1)]3^{a_7}}{\prod(a_i!)(a_2+1)9^r} && (2)\\
&=\sum_{A_3}\frac{(r+1)!(a_5-a_2)3^{a_7}}{\prod(a_i!)9^r} && (3)\\
&<\sum_{A_4}\frac{(r+1)!k5^b}{a_2!a_6!(a_2+k)!(a_6+k)!b!9^r} && (4)\\
&=\sum_{A_5}\frac{\frac14(r+1-b)(r+1)!5^b}{(r+1-b)!b!9^r}\left(\frac{r+1-b}{\frac12(r+1-b)}\right)^2 && (5)\\
&<\frac{9}{2\pi}\sum_{A_5}\binom{r+1}{b}\left(\frac59\right)^b\left(\frac49\right)^{r+1-b} && (6)\\
&\sim\frac{9}{4\pi}. && (7)
\end{aligned}$$

Three auxiliary lemmas will be used in verifying the preceding sequence. We shall prove these before continuing with the proof that $(r+1)[Q_{2r}(2,\geq 0)-Q_{2r}(-2,\geq 0)]\leq 9/(4\pi)$.

Lemma 11. *Let B be the set of all nonnegative integral (n_1,n_2,n_3) for which*

$n_1+n_2+n_3=n$, where n is a nonnegative integer. Then

$$\sum_B \frac{n_3}{n_1!n_2!(n_1+n_3)!(n_2+n_3)!} = \frac{\frac{1}{2}n}{(n!)^2}\binom{2n}{n}.$$

Lemma 12. *$\binom{2n}{n} < 2^{2n}/\sqrt{\pi n}$ for every $n \in \{1, 2, \ldots\}$.*

Lemma 13. *Let E_n be the set of all nonnegative even integers that do not exceed n, and let λ be strictly between 0 and 1. Then*

$$\sum_{j\in E_n} \binom{n}{j}\lambda^j(1-\lambda)^{n-j} \sim \frac{1}{2}.$$

Proof of Lemma 11. The identity

$$\sum \binom{n}{x}\binom{n}{y} = \binom{2n}{x+y},$$

where the sum is over all nonnegative integral (x, y) such that $x+y$ is constant, will be used in going from line three to line four in the ensuing sequence. The foregoing identity is noted, for example, by Feller [4, p. 62].

$$\begin{aligned}
&(n!)^2 \sum_B \frac{n_3}{n_1!n_2!(n_1+n_3)!(n_2+n_3)!} \\
&= \sum_B n_3 \binom{n}{n_1}\binom{n}{n_2} = \sum_{n_3=0}^{n} n_3 \sum_{\{(n_1,n_2):n_1+n_2=n-n_3\}} \binom{n}{n_1}\binom{n}{n_2} \\
&= \sum_{n_3=0}^{n} n_3 \binom{2n}{n-n_3} = \sum_{n_3=0}^{n} (n+n_3-n)\binom{2n}{n+n_3} \\
&= 2n \sum_{n_3=0}^{n} \binom{2n-1}{n-n_3} - n \sum_{n_3=0}^{n} \binom{2n}{n-n_3} \\
&= 2n\left[\tfrac{1}{2}(2^{2n-1}) + \binom{2n-1}{n}\right] - n\left[\frac{1}{2}\binom{2n}{n} + 2^{2n-1}\right] \\
&= \tfrac{1}{2}n\left[4\binom{2n-1}{n} - \binom{2n}{n}\right] = \tfrac{1}{2}n\binom{2n}{n}. \quad \square
\end{aligned}$$

Proof of Lemma 12. This follows easily from the bounds on Stirling's approximation for $n!$ given, for example, on p. 52 in Feller [4]. □

Proof of Lemma 13. Let $p(n)$ be the noted sum over E_n in Lemma 13. Then

$$p(n+1) = (1-\lambda)p(n) + \lambda[1-p(n)].$$

Figuratively speaking, the probability of an even number of successes in $n+1$ Bernoulli trials with success probability λ equals the probability of an even number of successes in the first n trials times the probability of failure at trial $n+1$, plus the probability of an odd number of successes in the first n trials times the probability of success at trial $n+1$. Since $0<\lambda<1$, $p(n)$ approaches the stable p that is the solution to $p=(1-\lambda)p+\lambda(1-p)$, which is $p=\frac{1}{2}$. □

We now return to sequence (2) through (7). Given $a=(a_1,\ldots,a_7)$, let a' and a^* be defined by

$$a_2'=a_2+1,\quad a_5'=a_5-1,\quad a_j'=a_j \quad \text{for } j\notin\{2,5\};$$
$$a_1^*=a_6,\quad a_6^*=a_1;\quad a_2^*=a_5,\quad a_5^*=a_2;\quad a_j^*=a_j \quad \text{for } j\in\{3,4,7\}.$$

To go from $(r+1)[Q_{2r}(2,\geq 0)-Q_{2r}(-2,\geq 0)]$ to (2), note first that all vectors in A_1 are uniquely generated from the vectors in A_2 that have $a_5>0$ by the mapping $a\to a'$ except for those that have $a_2'=0$. However, since

$$(r+1)\sum_{A_1\cup\{a_2=0\}}\frac{r!}{\prod(a_i!)}\left(\frac{3^{a_7}}{9^r}\right)=(r+1)(\tfrac{8}{9})^r\sum_{A_1\cup\{a_2=0\}}\frac{r!}{\prod(a_i!)}\left(\frac{3}{8}\right)^{a_7}\left(\frac{1}{8}\right)^{r-a_7}$$
$$\leq(r+1)(\tfrac{8}{9})^r,$$

and since $(r+1)(\frac{8}{9})^r\sim 0$ by l'Hospital's rule, the omission from A_1 of vectors that have $a_2=0$ will not affect the large r behavior of $(r+1)Q_{2r}(2,\geq 0)$. In like manner the $a_5=0$ vectors can be omitted from $(r+1)Q_{2r}(-2,\geq 0)$ so that

$$(r+1)[Q_{2r}(2,\geq 0)-Q_{2r}(-2,\geq 0)]\sim\sum_{A_2\cup\{a_5>0\}}\frac{(r+1)!}{\prod(a_i!)}\left[\frac{a_5}{a_2+1}-1\right]\frac{3^{a_7}}{9^r},$$

which verifies (2).

To obtain (3) from (2), first replace a_2+1 in (2) by $a_2''=a_2+1$, then drop the double prime to get (2) equal to

$$\sum_{A_{30}}\frac{(r+1)!(a_5-a_2)3^{a_7}}{\prod(a_i!)9^r}$$

with

$$A_{30}=\{(a_1,\ldots,a_7):\ a_1+a_2=a_5+a_6,\ a_1+a_3\geq a_4+a_6,$$
$$a_5>0,\ a_2>0,\ \textstyle\sum a_i=r+1\}.$$

Given $a\in A_{30}$, the mapping $a\to a^*$ gives $a^*\in A_{30}$ if and only if $a_6+a_3\geq a_4+a_1$, and if $a, a^*\in A_{30}$, then their terms in the A_{30} sum cancel since $a_5^*-a_2^*=$

$-(a_5-a_2)$. After all such cancellations, A_{30} retains only those a for which $a_6+a_3<a_4+a_1$, and it is these vectors that comprise A_3. Hence (3) equals (2).

The inequalities $a_1+a_3 \geq a_4+a_6$ and $a_3+a_6<a_1+a_4$ in A_3 imply $a_1>a_6$. Let $k=a_1-a_6$. Since $a_5-a_2=a_1-a_6$ in A_3, (3) equals

$$\sum_{A_{31}} \frac{(r+1)!k3^{a_7}}{a_2!a_6!(a_2+k)!(a_6+k)!a_3!a_4!a_7!9^r}$$

with

$$A_{31}=\{(a_2, a_6, k, a_3, a_4, a_7): k>0, a_2>0, a_4-a_3 \leq k, a_3-a_4<k,$$
$$2(a_2+a_6+k)+(a_3+a_4+a_7)=r+1\}.$$

The sum of $3^{a_7}/(a_3!a_4!a_7!)$ over all (a_3, a_4, a_7) for which $a_3+a_4+a_7=b$ is $5^b/b!$. Since this sum ignores the inequality constraints in A_{31}, and since A_4 is just A_{31} minus its inequality constraints and with $b=a_3+a_4+a_7$, it follows that (3) is less than (4).

In A_4, b is feasible if and only if $b \geq 0$ and $\frac{1}{2}(r+1-b)$ is an integer, in which case $r+1-b$ must be even. With b feasible and fixed, we next sum the terms in (4) that involve k, a_2 and a_6 over all (a_2, a_6, k) for which $a_2+a_6+k=\frac{1}{2}(r+1-b)$. By Lemma 11, this sum is

$$\sum_{\{a_2+a_6+k=\frac{1}{2}(r+1-b)\}} \frac{k}{a_2!a_6!(a_2+k)!(a_6+k)!} =$$
$$=\frac{[\frac{1}{4}(r+1-b)]}{\{[\frac{1}{2}(r+1-b)]!\}^2}\binom{r+1-b}{\frac{1}{2}(r+1-b)}.$$

When this is taken account of in (4), it follows that (4) equals (5).

The term for $b=r+1$ or $r+1-b=0$ in (5) equals zero. In addition, when $r+1-b$ is positive and even, Lemma 12 gives

$$\binom{r+1-b}{\frac{1}{2}(r+1-b)}^2<\frac{4^{r+1-b}}{\frac{1}{2}\pi(r+1-b)}.$$

When the right-hand side of this inequality is substituted into (5) in place of its left-hand side, we obtain (6) so that (5) is less than (6).

Finally, Lemma 13 implies that

$$\sum_{A_5}\binom{r+1}{b}\left(\frac{5}{9}\right)^b\left(\frac{4}{9}\right)^{r+1-b} \sim \frac{1}{2}$$

and therefore (6) $\sim 9/(4\pi)$. The sequence of (2) through (7) thus shows that $(r+1)[Q_{2r}(2, \geq 0)-Q_{2r}(-2, \geq 0)] \leq 9/(4\pi)$, and the proof of Lemma 10 is complete.

6. Additional results for odd n

It seems apparent from preceding sections that Kelly's conjectures are more amenable to resolution for odd n than for even n. Therefore, in attempting to extend the results derived above, it would appear more profitable to work initially with odd n. The main results we have obtained for odd n are Theorem 1, which says that

$$C(m,3) > C(m+1,3) \quad \text{for all } m \in \{2,3,\ldots\}, \tag{8}$$

and Theorem 2, which says that

$$C(3,n) > C(3,n+2) \quad \text{for all odd } n \geqslant 1. \tag{9}$$

In concluding this study we note several related results for odd n that arise from the following recursion relations in Gehrlein and Fishburn [6]:

$$C(4,n) = 2C(3,n) - 1 \quad \text{for odd } n \geqslant 1 \tag{10}$$

$$C(6,n) = 3 - 5C(3,n) + 3C(5,n) \quad \text{for odd } n \geqslant 1. \tag{11}$$

Theorem 6 follows the spirit of (8), while Theorem 7 is in the mode of (9).

Theorem 6. *The following hold for all odd* $n \geqslant 3$:
(a) $C(3,n) > C(4,n)$;
(b) $C(3,n) > C(5,n)$;
(c) $C(5,n) > C(6,n) \Rightarrow C(4,n) > C(5,n)$;
(d) $C(3,n) > C(6,n)$ *if and only if* $C(4,n) > C(5,n)$.

Proof. (a): By (10), $C(4,n) - C(3,n) = C(3,n) - 1 < 0$.
(b): If $C(5,n) \geqslant C(3,n)$, then (11) implies that $C(6,n) \geqslant 1 + 2[1 - C(5,n)]$, which is false since $1 + 2[1 - C(5,n)] > 1$.
(c): By (11), $C(3,n) = 3 - 4C(3,n) + 3C(5,n) - C(6,n)$. Since

$$1 > C(3,n), \qquad -2 + 4C(3,n) - 2C(5,n) > C(5,n) - C(6,n).$$

But, by (10), $-2 + 4C(3,n) = 2C(4,n)$. Therefore

$$2[C(4,n) - C(5,n)] > C(5,n) - C(6,n).$$

(d): By (10) and (11), $C(6,n) - C(3,n) = 3[C(5,n) - C(4,n)]$. □

Theorem 7. *The following hold for all odd* $n \geqslant 1$:
(a) $C(4,n) > C(4,n+2)$;
(b) $C(6,n) > C(6,n+2) \Rightarrow C(5,n) > C(5,n+2)$.

Proof. (a): By (10), $C(4,n)-C(4,n+2)=2[C(3,n)-C(3,n+2)]$. Then, by (9), $C(4,n)>C(4,n+2)$.

(b): By (11),

$$3[C(5,n)-C(5,n+2)]=[C(6,n)-C(6,n+2)]+5[C(3,n)-C(3,n+2)].$$

Hence (9) and $C(6,n)>C(6,n+2)$ imply that $C(5,n)>C(5,n+2)$. □

Theorems 1 and 6 imply that if $C(5,n)>C(6,n)$ is true for all odd $n\geq 3$, then $C(3,n)>C(4,n)>C(5,n)>C(6,n)$ for all odd $n\geq 3$. Hence a reasonable next step for Conjecture 1 would be to try to establish $C(5,n)>C(6,n)$ for odd $n\geq 3$. In like manner, it follows from Theorems 2 and 7 that if $C(6,n)>C(6,n+2)$ for odd $n\geq 1$, then $C(m,n)>C(m,n+2)$ for odd $n\geq 1$ and $m\in\{3,4,5,6\}$. Hence a reasonable next step for Conjecture 2 would be to try to establish $C(6,n)>C(6,n+2)$ for odd $n\geq 1$. Moreover, If this is true, then—as shown by our final theorem—it is true also that $C(5,n)>C(6,n)$ for odd $n\geq 3$.

Theorem 8. *If* $C(6,n)>C(6,n+2)$ *for odd* $n\geq 3$, *then* $C(5,n)>C(6,n)$ *for odd* $n\geq 3$.

Proof. Assume that $C(6\ n)>C(6,n+2)$ for odd $n\geq 3$ and, contrary to the theorem, suppose that $C(6,n)\geq C(5,n)$ for some odd $n\geq 3$. Then, by (11), $2C(5,n)\geq 5C(3,n)-3$. By (9) and Guilbaud's result [8, 9], $C(3,n)>0.912$, and therefore $C(6,n)\geq C(5,n)$ implies that $C(5,n)>0.78$. In order for this to be true, Theorem 7(b), the hypothesis that $C(6,n)>C(6,n+2)$, and known values [6] of $C(5,n)$ require $n\leq 7$. But values [6] of $C(5,n)$ and $C(6,n)$ give $C(5,n)>C(6,n)$ for $n\in\{3,5,7\}$. □

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