The Existence of Equilibrium with Price-Setting Firms

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Ever since Joseph Bertrand (1883), economists have been interested in static models of oligopoly where firms set prices. Francis Edgeworth's 1925 critique of Bertrand recognized, however, that, except in the case of constant marginal costs, there are serious equilibrium existence problems when firms produce a homogeneous good. In particular, Edgeworth proposed a modification of Bertrand's model in which firms have zero marginal cost up to some fixed capacity. He showed that, unless demand is highly elastic, price equilibrium may fail to exist.

Mixed strategies provide one way of avoiding this nonexistence problem, as various authors have noted. Martin Beckmann (1965), for instance, explicitly calculated mixed strategy equilibria in a symmetric example of the Bertrand-Edgeworth model. However, a general treatment of mixed strategies has suffered from the fact that the standard equilibrium existence lemmas (see, for example, K. Fan, 1952, and I. Glicksberg, 1952), require continuous payoff functions, whereas, in price-setting oligopoly, payoffs are inherently discontinuous—the firm charging the lowest price captures the whole market.

Recently, Partha Dasgupta and I (1986) and Leo Simon (1984) developed several existence theorems for discontinuous games. Dasgupta and I used one of the theorems to establish the general existence of mixed strategy equilibrium in the Bertrand-Edgeworth model when market demand as a function of price is continuous, downward sloping, and equal to zero for a sufficiently high price. In their study of Bertrand-Edgeworth competition in large economies, Beth Allen and Martin Hellwig (1983) extended this result to demand curves that do not necessarily intersect the horizontal axis and need not slope downward.

There have been several treatments of cost functions more general than the Bertrand-Edgeworth variety. R. Gertner (1985) established the existence of symmetric equilibrium in a model where firms are identical, have convex or concave costs, and choose output levels at the same time as prices. Also in a model of identical firms, H. Dixon (1984) proved existence when firms have convex costs and produce to order, that is, produce *after* other firms' prices are realized.

In this paper, I present some existence results that do not require symmetry and permit fairly general cost functions. These findings pertain both to the simultaneous choice of price and production level and to the formulation where a firm's output is set only after it knows others' prices. Existence in the former case is proved by direct application of the Dasgupta-Maskin/Simon theorems, as are the existence results mentioned above. The latter case, however, requires some additional argument. I give a sketch of this argument below; the details and more general results can be found in Dixon and myself (1986).

I. The Model

For simplicity, I shall consider only two firms; all results generalize immediately to any finite number. Firms produce the same good, and firm i, $i=1,2$, has total cost function $c_i(x_i)$, where x_i is the firm's output

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level. We assume

(1) $c_i(x)$ is continuous and nondecreasing; $c_i(0)=0$.

Firms face an industry demand curve $F: R_+ \to R_+$, where

(2) F is continuous with $F(0)=K$ for some $K>0$; $pF(p)-c_i(F(p)$ is maximized at $\bar{p}_i>0$, and, if there are multiple maximizers, $\bar{p}_i$ is the largest. Let $\bar{p}=\max p_i$.

Firm i may have a constraint K_i on its production level. In view of (2), we may assume, without loss of generality, that $K_i \leq K$.

Firm i's strategy consists of choosing a price $p_i \in [0, \bar{p}]$ and supply $s_i \in [0, K_i]$. Given the firms' strategies the demand facing firm i is

(3) $$d_i(p_1, s_1, p_2, s_2) = \begin{cases} F(p_i), & \text{if } p_i < p_j \\ G_i(p, s_1, s_2), & \text{if } p_1 = p_2 = p \\ H_i(p_1, p_2, s_j), & \text{if } p_i > p_j, \end{cases}$$

where G_i is a function such that

(4) $G_1(p, s_1, s_2) + G_2(p, s_1, s_2) = F(p)$; if $s_i > 0, G_i > 0$; if $s_i \geq s_j, G_i \geq G_j$; and $\min\{G_i(p, s_1, s_2), s_i\}$ is continuous in s_i,

and $H_i(p_1, p_2, s_j)$ is a function such that

(5) $$H_i(p_1, p_2, s_j) \leq F(p_i);\ H_i(p, p, s_j) = F(p) - s_j;\ \text{and } H_i \text{ is continuous.}$$

The first line of (3) simply posits that the firm charging the lower price attracts the entire market demand.

The second line stipulates that, if firms charge the same demand, they split the market in some way that depends on their supplies. Condition (4) tells us that a firm's share is a nondecreasing function of its supply and that a positive supply implies a positive share. Two examples of G_i's satisfying (4) are

$$G_i(p, s_1, s_2) = \begin{cases} \dfrac{s_i}{s_1+s_2} F(p), & \text{if } s_1 + s_2 > 0 \\ \dfrac{1}{2} F(p), & \text{if } s_1 + s_2 = 0 \end{cases}$$

and $G_i(p, s_1, s_2) = \alpha_i F(p)$, where $\alpha_1 + \alpha_2 = 1$.

The third line of (3) requires that the firm charging the higher price get less than full market demand. Moreover, if the two firms are charging approximately the same price, the demand facing the high-price firm is approximately full market demand at that price minus the supply of the other firm. Two examples of H_i's satisfying (5) are $H_i(p_1, p_2, s_j) = \max\{F(p_i) - s_j, 0\}$ and

$$H_i(p_1, p_2, s_j) = \max\left\{\frac{F(p_j) - s_j}{F(p_j)}, 0\right\} F(p_i).$$

The former rule is called parallel rationing (see Richard Levitan and Martin Shubik, 1972), whereas the latter is proportional rationing (see Edgeworth).

In the case where a firm sets production at the same time as price (production in advance), firm i's payoff is

(6) $$p_i x_i - c_i(s_i),$$

where

(7) $$x_i = \min\{s_i, d_i(p_1, s_1, p_2, s_2)\},$$

whereas in the case where it produces to order, the firm's payoff is

(8) $$p_i x_i - c_i(x_i).$$

II. Equilibrium in Discontinuous Games

Let us present a special case of the main existence theorems in Dasgupta's and my article, and in Simon. For $i=1,2$, let A_i be a convex, compact subset of R^2. The set A_i is player i's strategy space. Let $U_i: A_1 \times A_2 \to R$, the payoff function for player i, be (a) bounded and (b) continuous except at points $(p_1, s_1, p_2, s_2) \in A_1 \times A_2$ where $p_1 = p_2$. Assume, furthermore, that U_i is weakly lower semicontinuous in (p_i, s_i). That is, for any (p_1, s_1) there exists a sequence $\{(p_1^n, s_1^n)\}$ converging to (p_1, s_1) such that no two p_1^n's and no two s_1^n's are the same and such that, for any (p_2, s_2),

$$\text{(9)} \quad \lim_{(p_1^n, s_1^n) \to (p_1, s_1)} U_1(p_1^n, s_1^n, p_2, s_2) \geq U_1(p_1, s_1, p_2, s_2),$$

and similarly for player 2. Finally, suppose that $\Sigma_{i=1}^2 U_i$ is upper semicontinuous.

PROPOSITION: *Given the stated assumptions, a mixed strategy equilibrium exists. That is, there exist probability measures* (μ_1^*, μ_2^*) *such that*

$$\int U_1(p_1, s_1, p_2, s_2) d\mu_1^*(p_1, s_1) \times d\mu_2^*(p_2, s_2) \geq \int U_1(p_1, s_1, p_2, s_2)\, d\mu_1(p_1, s_1) \times d\mu_2^*(p_2, s_2) \qquad \textit{for all } \mu_1 \textit{ on } A_1;$$

$$\int U_2(p_1, s_1, p_2, s_2)\, d\mu_1^* \times d\mu_2^* \geq \int U_2(p_1, s_1, p_2, s_2)\, d\mu_1^* \times d\mu_2, \qquad \textit{for all } \mu_2 \textit{ on } A_2.$$

III. Equilibrium in Oligopoly

Let us take $A_i = [0, \bar{p}] \times [0, K_i]$ in our oligopoly model. In the case of production in advance, we can readily verify that firms' payoffs satisfy the hypotheses of the above proposition. From (2), profits are clearly bounded. Because F, the H_i's and the c_i's are continuous, profits are continuous except where $p_1 = p_2$. If $p_1 = 0$ or $s_1 = 0$, then U_1 is continuous at (p, s_1, p_2, s_2), and so (9) holds automatically. If p_1 and s_1 are both positive, then $d_1(p_1, s_1, p_2, s_2)$ is lower semicontinuous from the left. That is, at points of discontinuity (where $p_1 = p_2$), firm 1's demand jumps *downward* for a sequence $\{p_1^n\}$ of prices converging to p_1 from below. Hence, (9) holds for any convergent sequence $\{(p_1^n, s_1^n)\}$ where p_1^n converges to p_1 from below. Finally, observe that discontinuities simply entail a shift in demand from one firm to the other. Thus, although U_1 and U_2 are discontinuous, their sum is not. I conclude that the proposition applies.

THEOREM 1: *Given* (1)–(5), *a mixed strategy equilibrium exists in the case of production in advance* (*where firm i's profit is given by* (6)).

I turn next to production to order. Here we encounter a difficulty in the application of the proposition; namely, the sum of profits need no longer be continuous nor even upper semicontinuous. Although discontinuities still involve a shift in demand between firms, they now also entail a shift in *production*. Thus an increase in production by a less efficient firm at the expense of a more efficient one may induce a fall in total profit.

To deal with this difficulty, I modify the strategy spaces and payoff functions somewhat to restore upper semicontinuity. This will enable us to conclude that an equilibrium for the modified payoff functions exists. I then argue that the strategies for this equilibrium remain in equilibrium for the original payoff functions.

I first strengthen the assumptions about cost functions. In addition to (1), we require

$$\text{(10)} \qquad c_i(x) \text{ is strictly convex.}$$

For $i=1,2$ firm i's supply function is

$$s_i(p) = \arg \max_{x \in [0, K_i]} [px - c_i(x)].$$

From (1) and (10), $s_i(p)$ is well-defined, nondecreasing, and continuous. Notice that by assuming that costs are convex, we can conclude that $s_i = s_i(p)$ maximizes

$\min\{d_i, s_i\}p - c_i(\min\{d_i, s_i\})$, for any d_i. Accordingly, in the modified model, let firm i's strategy space be $\hat{A}_i = [0, \bar{p}]$. For any price p, let $V_i(p) = px_i - c_i(x_i)$ and $W_i(p) = py_i - c_i(y_i)$, where $x_i = \min\{s_i(p), F(p)\}$ and $y_i = \min\{s_i(p), \max\{F(p) - s_j(p), 0\}\}$.

Then,

$$(11)\quad W_1(p) \le U_1(p, s_1(p), p, s_2(p)) \le V_1(p);$$

$$(12)\quad W_2(p) \le U_2(p, s_1(p), p, s_2(p)) \le V_2(p).$$

The second inequality in (11) is strict if and only if the first inequality in (12) is strict. Moreover, the first inequality in (11) is strict if and only if the second inequality in (12) is strict. Thus we can choose $Z_i(p) \in [U_i(p, s_1(p), p, s_2(p)), V_i(p)]$ such that $Z_1(p)$ ($Z_2(p)$) is in the interior of the interval if the second inequality in (11) ((12)) is strict, and

$$(13)\quad Z_1(p) + Z_2(p) \ge \max\{W_1(p) + V_2(p), W_2(p) + V_1(p)\}.$$

If $\{(p_1^n, p_2^n)\}$ is a sequence converging to (p, p) then

$$(14)\quad \limsup \sum U_i(p_1^n, s_1(p_1^n), p_2^n, s_2(p_2^n)) \le \max\Big\{W_1(p) + V_2(p), W_2(p) + V_1(p), \sum U_i(p, s_1(p), p, s_2(p))\Big\}.$$

Define

$$(15)\quad \hat{U}_i(p_1, p_2) = \begin{cases} U_i(p_1, s_1(p_1), p_2, s_2(p_2)), & \text{if } p_1 \ne p_2 \\ Z_i(p), & \text{if } p_1 = p_2 = p \end{cases}$$

Combining (13)–(15), it can be deduced that $\Sigma\hat{U}_i$ is upper semicontinuous. If $p = 0$, $\hat{U}_i$ is continuous at p, and if $p > 0$, $\hat{U}_i$ is lower semicontinuous from the left. Hence, applying the above proposition (and simply ignoring the extra dimension of the strategy space), I conclude that the modified game has a mixed strategy equilibrium $(\hat{\mu}_1, \hat{\mu}_2)$.

I claim that $(\hat{\mu}_1, \hat{\mu}_2)$ also represents a mixed strategy equilibrium of the original game: if firm i plays p_i in the modified game, then it plays $(p_i, s_i(p_i))$ in the original game. Choose $\hat{p}_1$ in the support of $\hat{\mu}_1$. Then

$$(16)\quad \int \hat{U}_1(\hat{p}_1, p_2)\, d\hat{\mu}_2 \ge \int \hat{U}_1(p_1, p_2)\, d\hat{\mu}_2,$$

for all $p_1 \in [0, \bar{p}]$. If $\hat{\mu}_2$ places positive probability on $\hat{p}_1$ and $U_1(\hat{p}_1, s_1(\hat{p}_1), \hat{p}_1, s_2(\hat{p}_1)) \ne \hat{U}_1(\hat{p}_1, \hat{p}_2)$, then by (15) and from definition of $Z_1(\hat{p}_1)$ there exists $\tilde{p}_1$ slightly less than $\hat{p}_1$ such that $\int \hat{U}_1(\tilde{p}_1, p_2)\, d\hat{\mu}_2 > \int \hat{U}_1(\hat{p}_1, p_2)\, d\hat{\mu}_2$, a contradiction of (16). Hence, because $U_1(p_1, s_1(p_i), p_2, s_2(p_2))$ and $\hat{U}_1(p_1, p_2)$ differ only where $p_1 = p_2$, the left-hand side of (16) equals $\int U_1(\hat{p}_1, s_1(\hat{p}_1), p_2, s_2(p_2))\, d\hat{\mu}_2$. But, from (15), $\hat{U}_1(p_1, p_2)$ is greater than or equal to $U_1(p_1, s_1(p_1), p_2, s_2(p_2))$ everywhere, and so (16) implies

$$(17)\quad \int U_1(\hat{p}_1, s_1(\hat{p}_1), p_2, s_2(p_2))\, d\hat{\mu}_2 \ge \int U_1(p_1, s_1(p_1), p_2, s_2(p_2))\, d\hat{\mu}_2 \quad \text{for all } p_1.$$

Now suppose that for some p_1^0 and $s_1^0 \in [0, K_1]$,

$$\int U_1(\hat{p}_1, s_1(\hat{p}_1), p_2, s_2(p_2))\, d\hat{\mu}_2 < \int U_1(p_1^0, s_1^0, p_2, s_2(p_2))\, d\hat{\mu}_2.$$

Then, in view of (17), $U_1(p_1^0, s_1^0, p_1^0, s_2(p_1^0)) > U_1(p_1^0, s_1(p_1^0), p_1^0, s_2(p_1^0))$ and $\hat{\mu}_2(p_1^0) > 0$. Thus there exists $\tilde{p}_1$ slightly less than p_1^0 such that

$$\int U_1(\tilde{p}_1, s_1(\tilde{p}_1), p_2, s_2(p_2))\, d\hat{\mu}_2 > \int U_1(\hat{p}_1, s_1(\hat{p}_1), p_2, s_2(p_2))\, d\hat{\mu}_2,$$

a contradiction of (17). I conclude that

$$\int U_1(\hat{p}_1, s_1(\hat{p}_1), p_2, s_2(p_2))\, d\hat{\mu}_2 \geq \int U_1(p_1, s_1, p_2, s_2(p_2))\, d\hat{\mu}_2$$

for all p_1 and s_1.

I have demonstrated

THEOREM 2: *Given* (1)–(5) *and* (10), *a mixed strategy equilibrium exists in the case of production to order* (*where firm i's profit is given by* (8)).

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