

Efficient Renegotiation—Proof Equilibria in Repeated Games*

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We show that if a generic two-person finite game is repeated infinitely many times then, for discount factors sufficiently near 1, there exists a Pareto-efficient equilibrium that is weakly renegotiation-proof in the sense of J. Farrell and E. Maskin (1989, "Renegotiation in Repeated Games," *Games Econ. Behav.* 1, 327–360). One corollary of this proposition is a linear algebraic result that has some connection with Farkas's Lemma. © 1989 Academic Press, Inc.

In a companion paper (Farrell and Maskin, 1989), two definitions are offered for equilibrium strategies in an infinitely repeated game that are stable in the sense that the players will never agree to switch to other strategies during the course of play. A subgame-perfect equilibrium of the repeated game is said to be *weakly renegotiation-proof* (WRP) if, for any two continuation equilibria, neither Pareto-dominates the other. An equilibrium is *strongly renegotiation-proof* (SRP) if it is WRP and none of its continuation equilibria is Pareto-dominated by another WRP equilibrium. For more formal definitions, as well as for motivation and discussion of these concepts, the reader is referred to Farrell and Maskin.

In general, some or all of the continuation equilibria of a WRP equilibrium may be Pareto-inefficient. For example, the strategies in which the

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players always play the same Nash equilibrium of the one-shot game trivially form a WRP equilibrium, but they are Pareto-dominated by the analogous strategies corresponding to a better one-shot Nash equilibrium. Since much of the literature on repeated games presumes that players will choose Pareto-efficient payoffs if they can, one might ask whether every repeated game has efficient payoffs that are consistent with the constraint of renegotiation, provided that players' discount factors are close enough to 1.

The answer to this question is "no." Farrell and Maskin exhibit a game in which, regardless of the discount rate, the only WRP equilibrium is simply infinite repetition of the (unique) one-shot Nash equilibrium, and this Nash equilibrium is Pareto inefficient. The example, however, is unrobust in the sense that, if the payoffs are perturbed slightly, existence of a Pareto optimal WRP equilibrium is restored for discount factors near 1.

In this paper we prove that such existence is *generic* in two-player games. This means that the Farrell–Maskin example and others like it are not representative of two-person games in general. The result is used by Farrell and Maskin to establish conditions under which SRP equilibria exist. It also enables them to characterize the SRP equilibrium payoffs for discount factors near 1.

Before proving the theorem we introduce some notation (for the full complement of notation and definitions, see Farrell and Maskin). In the constituent one-shot game g , player i ($i = 1, 2$) has a finite set of pure actions $S_i = \{s_i^1, \dots, s_i^{n_i}\}$. The one-shot payoff in g from a typical pair of (pure or mixed) actions (a_1, a_2) is $g(a_1, a_2) \equiv (g_1(a_1, a_2), g_2(a_1, a_2))$ and V is the convex hull of the set of payoff pairs feasible in g . The proof of the theorem makes use of the fact (Theorem 1 in Farrell and Maskin) that a payoff pair $(v_1, v_2) \in V$ is WRP for some discount factor if either (v_1, v_2) corresponds to a Nash equilibrium of g or there exist action pairs (a_1^1, a_2^1) and (a_1^2, a_2^2) such that

$$\max_{s_2^j} g_2(a_1^2, s_2^j) < v_2, \tag{1}$$

$$\max_{s_1^i} g_1(s_1^i, a_2^1) < v_1, \tag{2}$$

$$g_1(a_1^2, a_2^2) \geq v_1, \tag{3}$$

and

$$g_2(a_1^1, a_2^1) \geq v_2. \tag{4}$$

A two-person $n_1 \times n_2$ game can be regarded as an $2n_1 \cdot n_2$ -dimensional vector. We shall say that a property of such games holds *generically* if it holds for an open and dense set of $2n_1 \cdot n_2$ -dimensional vectors.

THEOREM. *In a generic two-player $n_1 \times n_2$ game there exists a weakly renegotiation-proof equilibrium of the repeated game that is Pareto-efficient for discount factors near enough to 1. That is, for each game in an open and dense set there exists $\delta < 1$ such that for all $\delta \geq \delta$ there exists a WRP equilibrium of the repeated game with discount factor δ such that the equilibrium average payoffs lie on the Pareto frontier of V .*

Proof. Consider a two-person game g . The Pareto boundary of V (excluding horizontal or vertical segments except for, respectively, their right and upper endpoints) consists of a finite number of downward-sloping faces connected by vertices.¹ Given an arbitrary point (\bar{v}_1, \bar{v}_2) on the Pareto boundary of V (but excluding the endpoints), we can find a positive affine transformation (i.e., a normalization) \bar{g} of g such that, in the normalized game, this point is at the origin and all of V lies on or below the straight line of slope -1 through the origin and such that if $\bar{g}_i = \alpha_i g_i + \gamma_i$, $i = 1, 2$, then $\alpha_1 + \alpha_2 = 1$. For example, let (v_1^k, v_2^k) be a vertex of V belonging to the same face of V as (\bar{v}_1, \bar{v}_2) (it is always possible to choose such a vertex not equal to (\bar{v}_1, \bar{v}_2)). Then define \bar{g} by $\bar{g}_1 = (1/(\beta + 1))(g_1 - \bar{v}_1)$ and $\bar{g}_2 = (1/(\beta + 1))(\beta g_2 - \beta \bar{v}_2)$, where $\beta = (\bar{v}_1 - v_1^k)/(v_2^k - \bar{v}_2) > 0$.

The set of such normalizations of g can be mapped homeomorphically onto the interval $(0, 1)$, where points near 0 correspond to transformations of the game in which $(0, 0)$ is near player 2's best and player 1's worst Pareto-efficient payoff and points near 1 correspond to those in which $(0, 0)$ is near player 2's worst and player 1's best Pareto-efficient payoff. Continuously moving through $(0, 1)$ is equivalent, under this homeomorphism, to continuously changing the normalization so that the origin traces out the whole Pareto boundary, always keeping V on or below the line $x + y = 0$. Indeed, the homeomorphism can be extended to the closed interval $[0, 1]$ by identifying 0 with the limit of a sequence of normalizations that transform points closer and closer to player 2's best Pareto-efficient payoff to the origin, with analogous treatment for the endpoint 1.² Define the correspondence $\Phi: [0, 1] \rightarrow \mathbf{R}$ as follows. $\Phi(t)$ is the set of numbers α such that, with the normalization t , there exist mixed actions (a_1, a_2) such that

$$\max_{s_2^j} g_2(a_1, s_2^j) \leq \alpha \quad (5)$$

¹ We can assume that there are at least two such vertices, because, if there were only one, it would correspond to a Pareto-efficient Nash equilibrium of g , and the theorem would be proved.

² We refrained from defining the homeomorphism onto the *closed* interval initially because the set of normalizations (satisfying our stipulated properties) corresponding to player 1's best Pareto optimum is not closed (nor is that corresponding to player 2's best Pareto optimum). By starting with $(0, 1)$, we include only those transformations corresponding to 0 that are limits of transformations corresponding to points in the open interval, and this set is closed (so is the analogous set corresponding to 1).

$$\max_{s_1^i} g_1(s_1^i, a_2) \leq -\alpha \tag{6}$$

and, for some (s_1^i, s_2^j) ,

$$g_1(a_1, s_2^j) \geq -\alpha \tag{7}$$

$$g_2(s_1^i, a_2) \geq \alpha. \tag{8}$$

We emphasize that, in (5)–(8), g has been normalized (according to t). We show that Φ is nonempty, convex-valued, and upper hemicontinuous and that $\Phi(1) \subseteq \mathbf{R}_+$ and $\Phi(0) \subseteq \mathbf{R}_-$, where \mathbf{R}_+ and \mathbf{R}_- denote, respectively the nonnegative and nonpositive real numbers. This implies that, for some $t \in [0, 1]$, $0 \in \Phi(t)$ and (since $(0, 0)$ is on the Pareto boundary when the game is normalized in the corresponding way) that the weak versions of inequalities (1)–(4) hold for some Pareto-efficient pair (v_1, v_2) .

NONEMPTINESS OF $\Phi(t)$

With the normalization as described above, $g_1(s_1^i, s_2^j) + g_2(s_1^i, s_2^j) \leq 0$ for all (s_1^i, s_2^j) with equality for at least one pair (s_1^{i*}, s_2^{j*}) (and, generically, for at most two pairs). Define a new game \hat{g} as follows:

$$\left. \begin{aligned} \hat{g}_1(s_1^i, s_2^j) &= g_1(s_1^i, s_2^j) \\ \hat{g}_2(s_1^i, s_2^j) &= -\hat{g}_1(s_1^i, s_2^j) \end{aligned} \right\} \text{if } j = j^*$$

$$\left. \begin{aligned} \hat{g}_2(s_1^i, s_2^j) &= g_2(s_1^i, s_2^j) \\ \hat{g}_1(s_1^i, s_2^j) &= -\hat{g}_2(s_1^i, s_2^j) \end{aligned} \right\} \text{if } i = i^*.$$

If $i \neq i^*, j \neq j^*$, increase (if necessary) either or both of the players' payoffs in such a way that they sum to zero. For example, set $\hat{g}_1(s_1^i, s_2^j) = g_1(s_1^i, s_2^j)$ and $\hat{g}_2(s_1^i, s_2^j) = -\hat{g}_1(s_1^i, s_2^j)$. Then \hat{g} is a zero-sum game and $\hat{g}(s_1^i, s_2^j) \geq g(s_1^i, s_2^j)$ for all (s_1^i, s_2^j) . Let the value of \hat{g} be $-\alpha$ and let (a_1, a_2) be a pair of minimax actions. Then

$$\max_{s_2^j} g_2(a_1, s_2^j) \leq \max_{s_2^j} \hat{g}_2(a_1, s_2^j) = \alpha$$

$$\max_{s_1^i} g_1(s_1^i, a_2) \leq \max_{s_1^i} \hat{g}_1(s_1^i, a_2) = -\alpha$$

$$g_1(a_1, s_2^{j*}) = \hat{g}_1(a_1, s_2^{j*}) \geq -\alpha$$

and

$$g_2(s_1^{i*}, a_2) = \hat{g}_2(s_1^{i*}, a_2) \geq \alpha.$$

CONVEXITY OF $\Phi(t)$

Let $\alpha_1, \alpha_2 \in \Phi(t)$, where $\alpha_1 < \alpha_2$ and take $\alpha_3 = \lambda\alpha_1 + (1 - \lambda)\alpha_2$ for some $\lambda \in (0, 1)$. Also, let $a_1^1, a_2^1, s_1^1, s_2^1$ be the actions that satisfy (5)–(8) for α_1 and let the corresponding actions for α_2 be $a_1^2, a_2^2, s_1^2, s_2^2$. Then

$$\begin{aligned} \max_{s_2^j} g_2(a_1^j, s_2^j) &\leq \alpha_1 < \alpha_3 \\ \max_{s_1^i} g_1(s_1^i, a_2^i) &\leq -\alpha_2 < -\alpha_3 \\ g_1(a_1^1, s_2^1) &\geq -\alpha_1 > -\alpha_3 \\ g_2(s_1^2, a_2^2) &\geq \alpha_2 > \alpha_3. \end{aligned}$$

Therefore $\alpha_3 \in \Phi(t)$.

UPPER HEMICONTINUITY OF Φ

Let $\{t^q\}$ and $\{\alpha^q\}$ be sequences such that $t^q \rightarrow t^0$, $\alpha^q \in \Phi(t^q)$, and $\alpha^q \rightarrow \alpha^0$, and let $a_1^q, a_2^q, s_1^q, s_2^q$ be actions that satisfy (5)–(8) for α^q , given the normalization t^q . The sequence $\{a_1^q, a_2^q, s_1^q, s_2^q\}$ has a convergent subsequence; call its limit $a_1^0, a_2^0, s_1^0, s_2^0$. By continuity these actions must satisfy (5)–(8) for α^0 , with normalization t^0 . Hence $\alpha^0 \in \Phi(t^0)$ and Φ is upper hemicontinuous.

Consider the normalization corresponding to $t = 0$. In that case

$$\max_{s_1^i, s_2^j} g_2(s_1^i, s_2^j) = 0.$$

Therefore, if $\alpha \in \Phi(0)$, $\alpha \leq 0$ by (8). Similarly, if $t = 1$,

$$\max_{s_1^i, s_2^j} g_1(s_1^i, s_2^j) = 0;$$

so, if $\alpha \in \Phi(1)$, $\alpha \geq 0$ by (7). This concludes the demonstration that, for some Pareto-efficient payoff pair (v_1, v_2) , (1)–(4) hold weakly.

The remainder of the proof is concerned with showing that, generically, either there exists a Pareto-efficient Nash equilibrium or else (1) and (2) can be made to hold strictly, as required. Let t_0 be a normalization such that $0 \in \Phi(t_0)$. Suppose first that $\Phi(t_0)$ is a singleton. Then every zero-sum game \hat{g} obtained from g (with normalization t_0) by transformations of the type described above has value zero. Let (s_1^{i*}, s_2^{j*}) be a pure action pair such that (given this normalization) $g_1(s_1^{i*}, s_2^{j*}) + g_2(s_1^{i*}, s_2^{j*}) = 0$ and $g_2(s_1^{i*}, s_2^{j*}) \geq 0$ (such a pair exists because the boundary of V^* passes through the origin). If there is a second pure action pair with correspond-

ing payoffs that sum to zero denote this by (s_1^{i**}, s_2^{j**}) ; clearly we can assume that $g_2(s_1^{i**}, s_2^{j**}) \leq 0$ (again, because the boundary of V^* passes through the origin). Consider the zero-sum game \hat{g} in which $\hat{g}_1(s_1^i, s_2^j) = g_1(s_1^i, s_2^j)$ unless $i = i^*$, in which case $\hat{g}_2(s_1^i, s_2^j) = g_2(s_1^i, s_2^j)$, and let (a_1, a_2) be minimax actions for \hat{g} . Note that, generically, the transformation from g to \hat{g} produces a strict increase in player 2's payoffs except when player 1 plays $s_1^{i^*}$ or when the two players jointly play (s_1^{i**}, s_2^{j**}) (since, generically, there are not more than two pure action pairs whose payoffs sum to zero). Now, $g_1(a_1, s_2^{j^*}) = \hat{g}_1(a_1, s_2^{j^*}) \geq 0$, and so (3) holds for $(a_1^j, a_2^j) = (a_1, s_2^{j^*})$ and $(v_1, v_2) = (0, 0)$. Moreover, if a_1 puts nonzero probability weight on actions other than $s_1^{i^*}$ and s_1^{i**} , then

$$\max_{s_2^j} g_2(a_1, s_2^j) < \max_{s_1^i} \hat{g}_2(a_1, s_2^j) = 0,$$

and so (1) indeed holds strictly.

If a_1 puts all weight on $s_1^{i^*}$ and s_1^{i**} the strict inequality might not hold; we distinguish among three possible cases and consider them in turn.

(i) $a_1 = s_1^{i^*}$. Then, since, by assumption, $g_2(s_1^{i^*}, s_2^{j^*}) \geq 0$ and since the value of \hat{g} is zero, $g_2(s_1^{i^*}, s_2^{j^*}) = \hat{g}_2(s_1^{i^*}, s_2^{j^*}) = 0$ and, for all s_2^j , $g_2(s_1^{i^*}, s_2^j) = \hat{g}_2(s_1^{i^*}, s_2^j) \leq 0$. Generically, therefore, $a_2 = s_2^{j^*}$ and $(s_1^{i^*}, s_2^{j^*})$ is then a Pareto-efficient Nash equilibrium of g (since $g_1(s_1^{i^*}, s_2^{j^*}) = \hat{g}_1(s_1^{i^*}, s_2^{j^*})$, which proves the theorem.

(ii) $a_1 = s_1^{i**} \neq s_1^{i^*}$. Let $s_2^{j^0}$ solve $\max_{s_2^j} g_2(a_1, s_2^j)$. If $s_2^{j^0} \neq s_2^{j**}$, then $g_2(s_1^{i**}, s_2^{j^0}) < \hat{g}_2(s_1^{i**}, s_2^{j^0}) \leq 0$, and so (1) holds if we take $a_2^j = s_1^{i**}$. Assume therefore that $s_2^{j^0} = s_2^{j**}$. Now $g_2(s_1^{i**}, s_2^{j**}) < 0$ again implies that (1) holds. Thus suppose that $g_2(s_1^{i**}, s_2^{j**}) = 0$. Then, $g_1(s_1^{i**}, s_2^{j**}) = 0$ (since the payoffs for (s_1^{i**}, s_2^{j**}) sum to zero) and so, generically, $g_1(s_1^{i**}, s_2^j) \neq 0$ for all $j \neq j^{**}$. Hence, because $g_1(s_1^{i**}, a_2) = 0$ (since $\hat{g}_1(a_1, a_2) = 0$, and $\hat{g}_1(s^i, s^j) = g_1(s^i, s^j)$ if $i \neq i^*$), we must have $a_2 = s_2^{j**}$. But then (s_1^{i**}, s_2^{j**}) is a Pareto-efficient Nash equilibrium, completing the proof.

(iii) $a_1 = \lambda s_1^{i**} + (1 - \lambda)s_1^{i^*}$ for some $\lambda \in (0, 1)$. Again, let $s_2^{j^0}$ be a best response to a_1 in g . As in (ii) we are done if $j^0 \neq j^{**}$. Hence assume that $j^0 = j^{**}$. If $a_2 = s_2^{j**}$ then $\hat{g}_1(s_1^{i^*}, s_2^{j**}) = \hat{g}_1(s_1^{i**}, s_2^{j**}) = 0$; therefore, $\hat{g}(s_1^{i^*}, s_2^{j**}) = \hat{g}(s_1^{i**}, s_2^{j**}) = (0, 0)$ and hence (since $g_2(s_1^{i^*}, s_2^j) = \hat{g}_2(s_1^{i^*}, s_2^j)$ and $g_2(s_1^{i**}, s_2^{j**}) = \hat{g}_2(s_1^{i**}, s_2^{j**})$), $g_2(s_1^{i^*}, s_2^{j**}) = g_2(s_1^{i**}, s_2^{j**}) = 0$, which is nongeneric. If $a_2 \neq s_2^{j**}$ then there exists a pure action not equal to s_2^{j**} that is a best response to a_1 in \hat{g} . We claim that we can assume that this pure action is $s_2^{j^*}$. To see why, consider the following argument.

Perturb \hat{g} to obtain a new zero-sum game g' by slightly reducing player 2's payoffs (and increasing player 1's payoffs) unless $s_2^j = s_2^{j**}$ or $s_2^j = s_2^{j^*}$

or $s_1^i = s_1^{i*}$. This can be done in such a way that $g_2'(s_1^i, s_2^j) > g_2(s_1^i, s_2^j)$ unless $s_1^i = s_1^{i*}$ or $(s_1^i, s_2^j) = (s_1^{i**}, s_2^{j**})$. Since by assumption, $\{0\} = \Phi(t_0)$, the value of g' is zero. Let (a_1', a_2') be minimax actions for g' . If a_1' puts positive probability on an action other than s_1^{i*} or s_1^{i**} , we are done from the above argument. Similarly, if $a_1' = s_1^{i*}$ we are done. Assume therefore that $a_1' = \gamma s_1^{i*} + (1 - \gamma)s_1^{i**}$ for some $\gamma \in [0, 1)$. If there is a best response to a_1' in g that is not s_2^{j**} , we are done, as above. Thus suppose that s_2^{j**} is the unique best response to a_1' . If $g_2(a_1', s_2^{j**}) < 0$, then the proof is complete. Hence (since $g_2 = g_2'$ at (s_1^{i*}, s_2^{j**}) and (s_1^{i**}, s_2^{j**})) we can assume $\gamma g_2'(s_1^{i*}, s_2^{j**}) + (1 - \gamma)g_2(s_1^{i**}, s_2^{j**}) = \gamma g_2'(s_1^{i*}, s_2^{j**}) + (1 - \gamma)g_2'(s_1^{i**}, s_2^{j**}) = 0$. Also, by definition of g' , $\lambda g_2'(s_1^{i*}, s_2^{j**}) + (1 - \lambda)g_2(s_1^{i**}, s_2^{j**}) = \lambda \hat{g}_2(s_1^{i*}, s_2^{j**}) + (1 - \lambda)\hat{g}_2(s_1^{i**}, s_2^{j**}) = 0$ (since the value of \hat{g} is zero and s_2^{j**} is a best response to a_1 in \hat{g}). Hence since, generically, the value of g_2 is different at (s_1^{i*}, s_2^{j**}) and (s_1^{i**}, s_2^{j**}) , $\gamma = \lambda$ and so $a_1' = a_1$. Now, a best response to a_1 in g' must yield player 2 a payoff of zero. But the effect of the perturbation from \hat{g} to g' is to reduce 2's payoff against a_1 from all pure actions other than a_2^{j*} and a_2^{j**} . Hence, such actions must earn a payoff less than 0 in g' (since they earn at most 0 in \hat{g}). We conclude that the only possible pure-action best responses to a_1 in g' are a_2^{j*} and a_2^{j**} . Replacing \hat{g} by g' thus establishes our claim of the preceding paragraph.

Because s_2^{j*} is a best response to a_1 in \hat{g} , we have

$$\lambda \hat{g}_2(s_1^{i*}, s_2^{j*}) + (1 - \lambda)\hat{g}_2(s_1^{i**}, s_2^{j*}) = 0 \quad (9)$$

and therefore, since $\hat{g}_2(s_1^{i*}, s_2^{j*}) = g_2(s_1^{i*}, s_2^{j*}) \geq 0$, $\hat{g}_2(s_1^{i**}, s_2^{j*}) \leq 0$. But \hat{g} is zero-sum, so $\hat{g}_1(s_1^{i**}, s_2^{j*}) \geq 0$ and, by assumption, $\hat{g}_1(s_1^{i**}, s_2^{j**}) \geq 0$. Now s_1^{i**} gives player 1 a payoff of zero against a_2 , which is a mixture of s_2^{j**} and s_2^{j*} , putting positive probability on s_2^{j*} . Therefore $\hat{g}(s_1^{i**}, s_2^{j*}) = (0, 0)$, and so, from (9), $\hat{g}(s_1^{i*}, s_2^{j*}) = (0, 0)$. This means that $g_1(s_1^{i*}, s_2^{j*}) = g_1(s_1^{i**}, s_2^{j*}) = 0$, which is nongeneric.

We have shown that, if $\Phi(t_0)$ is a singleton, then, generically, either (1) and (3) hold or there exists a Pareto-efficient Nash equilibrium of g .

If $\Phi(t_0)$ is not a singleton, it must be a nondegenerate, closed interval from the convexity and upper hemicontinuity of Φ . If $0 \in \text{int } \Phi(t_0)$ then, for some $\varepsilon > 0$ there exist $\bar{a}_1, \bar{a}_2, \bar{s}_1^i, \bar{s}_2^j$ such that

$$\max_{s_2^j} g_2(\bar{a}_1, s_2^j) \leq -\varepsilon < 0$$

$$\max_{s_1^i} g_1(s_1^i, \bar{a}_2) \leq -\varepsilon < 0$$

$$g_1(\bar{a}_1, \bar{s}_2^j) \geq \varepsilon > 0$$

$$g_2(\bar{s}_1^i, \bar{a}_2) \geq \varepsilon > 0,$$

setting α equal to ε and $-\varepsilon$ in turn in (5)–(8). Therefore (5) and (6) hold strictly. Suppose, finally, that 0 is an endpoint of the interval $\Phi(t_0)$. Without loss of generality, take it to be the left endpoint. If $t_0 = 0$ then $g_2(s_1^i, s_2^j) \leq 0$ for all (s_1^i, s_2^j) , and so $\Phi(t_0)$ must be a singleton after all. Thus we may assume that $t_0 > 0$. If, given normalization t_0 , the first vertex of V to the “northwest” of the origin (which must exist since $t_0 > 0$) lies below the line $x + y = 0$, choose the maximum $t' < t_0$ such that, with *this* normalization, the vertex lies on the line $x + y = 0$. Since the games corresponding to the interval $[t', t_0)$ are all “rotations” of one another (i.e., they differ only in the scale of players’ payoffs), $0 \in \Phi(t_0)$ implies that $0 \in \Phi(t)$ for all $t \in [t', t_0]$. Also, by upper hemicontinuity and convex-valuedness of Φ , either (i) $0 \in \text{int } \Phi(t)$ for some $t \in [t', t_0]$; or (ii) $\Phi(t)$ is a singleton for some $t \in [t', t_0]$; or (iii) 0 is the left endpoint of $\Phi(t')$. In cases (i) and (ii) the theorem is proved. Assume therefore that (iii) holds and let g' be the normalization corresponding to t' . Because, given this normalization, the first vertex of V to the northwest of the origin lies on the line $x + y = 0$, we can choose $\varepsilon > 0$ such that the transformed game g'' , where $g''_1 = g'_1 + \varepsilon$ and $g''_2 = g'_2 - \varepsilon$ lie in the set of admissible normalizations. Then, if t'' is the point corresponding homeomorphically to g'' , we have $\Phi(t'') = \{\alpha - \varepsilon \mid \alpha \in \Phi(t)\}$ since any actions that satisfy (5)–(8) for α with the game g will satisfy (5)–(8) for $\alpha - \varepsilon$ with the game g'' . Therefore if 0 is the left endpoint of $\Phi(t')$, $0 \in \text{int } \Phi(t'')$ for some ε small enough. This concludes the proof of the theorem.

Q.E.D.

If we take g to be a symmetric game, the theorem has the following linear algebraic corollary. (\mathbf{R}_+^n is the set of n -vectors with nonnegative elements not all of which are zero.)

COROLLARY. *Consider an $n \times n$ matrix $A = (a_{ij})$ such that, for all i and j , $a_{ij} + a_{ji} \leq 0$, with equality for some choice of indices. There exist $x, y \in \mathbf{R}_+^n$ such that*

$$x^T A y \geq 0 \tag{10}$$

and

$$A x \leq 0, \tag{11}$$

where x^T denotes the transpose of x .

Proof. Let A and A^T be the payoff matrices for players 1 and 2, respectively, in a symmetric, two-person game. From symmetry and our normalization, the proof of the theorem implies that for $(v_1, v_2) = (0, 0)$ there

exists (a_1^2, a_2^2) satisfying (1) and (3) with weak inequalities. But if we take $x = a_1^2$ and $y = a_2^2$, (1) becomes (11) and (3) becomes (10). Q.E.D.

Although this result bears some resemblance to well-known linear algebraic propositions (e.g., Farkas's Lemma), it does not appear to be in the literature already. Moreover, it does not seem to follow directly from standard results.

REFERENCE

- FARRELL, J., AND MASKIN, E. (1989). "Renegotiation in Repeated Games," *Games Econ. Behav.* **1**, 327–360.