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# Notes, Comments, and Letters to the Editor

## On the Dispensability of Public Randomization in Discounted Repeated Games*

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We show that any feasible, individually rational payoffs of an infinitely repeated game can arise as subgame perfect equilibrium payoffs if the discount factor is close enough to one even if mixed strategies are not observable and public randomizations are not available. *Journal of Economic Literature* Classification Number: 026. 

### 1. INTRODUCTION

The Folk Theorem for repeated games asserts that any feasible, individually rational payoffs for a one-shot game can arise as Nash equilibrium average payoffs when the game is infinitely repeated and the players are sufficiently patient. In our paper [3], which examined when this result extends to subgame perfect equilibrium,[†] we assumed that the players can condition their play on the realization of a publicly observed random variable. We asserted, however, that abandoning the assumption would lead to only a slight weakening of the results; viz., that when a

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† See also [2], which considers average payoffs that Pareto dominate a one-shot Nash equilibrium.



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payoff vector can be attained by a perfect equilibrium with public randomization, it can be approximately attained by an equilibrium without public randomization, when there is sufficiently little discounting. This note shows that, in fact, the qualification "approximately" is not needed: Under the same hypotheses as in [3], all feasible individually rational payoffs can be exactly attained in perfect equilibrium without public randomization.

Although this stronger result is of some interest by itself, its true significance appears in connection with mixed strategies. Rubinstein's [6] proof of the perfect Folk Theorem in games with the zero discounting restricted players to pure strategies, or equivalently, assumed that a player's choice of a ***mixed*** strategy in any period is observable by his fellow players. The assumption of pure strategies is restrictive because typically the range of individually rational payoffs is greater when players are allowed to use mixed strategies to punish their opponents. The alternative hypothesis that a player's randomization are *ex* ***post*** observable is clearly restrictive as well.[1]

Section 6 of [3] showed how to extend the Folk Theorem to allow for mixed strategies when only a player's realized actions, and not his choices of randomizing probabilities, are observable. (Here and subsequently, "Folk Theorem" refers to the version with perfect equilibrium and discounting.) The key was the observation that a player can be induced to use a mixed strategy to minimax an opponent by making her continuation payoff depend on her current action in a way that renders her exactly indifferent among the various choices in the mixed strategy's support.

This extension relied on public randomization to ensure that any individually rational continuation payoffs can be exactly attained. If, without public randomization, the continuation payoffs could merely be approximated, a minimaxing player might not be exactly indifferent over the support of her mixed strategy, and our construction would fail. Thus, if we obtain only an approximate version of the Folk Theorem without public randomization, our construction cannot accommodate unobservable mixed strategies.

Attaining payoffs exactly is also essential for the argument in our paper [4], which provides sufficient conditions for the sets of Nash and perfect equilibrium payoffs to coincide for discount factors less than one.

[1] When punishment strategies are mixed, not all deviations from the prescribed punishment can be observed. Thus a more subtle construction is required to ensure that players will choose to carry out the prescribed punishments. However, when the players evaluate time streams by the limit of their average payoff, as in Aumann and Shapley [1], they are prepared to forego any deviation that increases their payoff in only finitely many periods. Since finite punishments suffice to enforce any individually rational path as a *Nash* equilibrium, the issue of detecting and deterring deviations from the punishment strategies does not arise under the time-average criterion (which implies zero discounting).

Although the results of that paper are first proved assuming public randomization, it is then shown, using our results here, that this assumption, as in [3], is unnecessary.

## 2. The Model

We consider a finite n-player game in normal form,

$$g: A \to \mathbb{R}^n,$$

where $A = A_1 \times \dots$ x $A$, and $A_i$ is player i's (finite) action space. Let $\Sigma_i$ be the set of player $i$'s mixed strategies, i.e., the probability distributions over $A_i$, and take $\Sigma = \Sigma_1$ x . . x $\Sigma_n$. To simplify notation, we will write g,(a) for player $i$'s payoff given the mixed strategy vector $\sigma \in \Sigma$.

In repeated versions of g, each player's probability mixture over actions at time $t$ can depend on the actions chosen at all previous times. More formally, let h(t) $\in A^{t-1} \equiv$ H(t) be the realized actions from time zero through time $t$ 1. Player $i$'s strategy is a sequence of maps (one for each period) from $H(t)$ to $\Sigma_i$. Note that, at any time $t$, player $i$'s strategy does not depend on the past randomizing probabilities of his opponents, but only on their realized actions.

In the infinitely repeated game $G_\delta$, each player $i$'s payoff is the discounted average $\pi_i$ of his per-period payoffs, with common discount factor $\delta$; that is,

$$\pi_i \equiv (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} g_i(\sigma(t)),$$

where $\sigma(t)$ is the probability distribution of actions chosen in period $t$.

The *continuation payoffs* at time $s$ associated with the sequence $\{a(t)\}$ are simply $(1-\delta) \sum_{t=s}^{\infty} \delta^{t-s} g(a(t))$, i.e., the discounted average of per-period payoffs starting at time s.

For each player $j$, choose "minimax strategies" $m^j = (m_1^j, \dots m_n^j)$ such that

$$m_{-j}^j \in \arg\min_{\sigma_{-j}} (\max_{a_j} g_j(a_j, \sigma_{-j})),$$

$$m_j^j \in \arg\max_{a_j} g_j(a_j, m_{-j}^j),$$

and

$$v_j^* = \max_{a_j} g_j(a_j, m_{-j}^j) = g_j(m^j).$$

(Here "$m^j_{-j}$" is a mixed strategy selection for players other than $j$, and $g_j(a_j, m^j_{-j}) = g_j(m^j_i, \ldots m^j_{j-1}, a_j, m^j_{j+1}, \ldots m^j_n)$). We call $v_j^*$ player j's reservation value. It is well known that player j's discounted average payoff must be at least $v_j^*$ in any equilibrium of g, whether or not g is repeated.

Henceforth we shall measure the payoffs of the game g so that $(v_1^*, \ldots, v_n^*) = (0, \ldots 0)$. Call $(0, \ldots 0)$ the *minimax* point. Take $\bar{v}_i = \max_a g_i(a)$. Moreover, let

$$U = \{(v_1, \ldots, v_n) \mid \text{there exists } a \in A \text{ with } g(a) = (v_1, \ldots, v_n)\},$$

$$V = \text{convex hull of } U,$$

and

$$V^* = \{(v_1, \ldots, v_n) \in V \mid v_i > 0 \text{ for all } i\}.$$

## 3. Generating Payoffs by Deterministic Sequences

When public randomizing devices are available, any point in $V$ can be attained by randomizing over points in $U$. For discount factors near 1, such a point can alternatively be generated by a deterministic sequence of actions $\{a(t)\}$, as in Lemma 1.[2] This is not sufficient to establish the Folk Theorem, however, because even if $v \in V^*$, the sequence $\{a(t)\}$ might have the property that the continuation payoffs beginning in some period $\tau$ do not belong to $V^*$. In that case, some player would prefer to deviate from the sequence, even if doing so caused his opponents to minimax him thereafter.

Accordingly, Lemma 2 establishes that, for every $v \in V^*$ and $\varepsilon > 0$, there is a $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$ there is a deterministic sequence of actions which yields average payoffs v and such that the continuation payoffs at each date are within $\varepsilon$ of v. Moreover, we show that $\underline{\delta}$ can be chosen so that it applies uniformly to all $v'$ for which $v'_i \geqslant \varepsilon$ for all $i$. Section 4 explains how these results allow us to dispense with public randomization in the proof of the Folk Theorem.

Lemma 1 has already been obtained by Sorin [7]. We restate the proof because we use the algorithm involved in our proof of Lemma 2. Write $A = \{a^1, \ldots a^m\}$ and, for each $k$, let $w^k = g(a^k)$. Thus, $\{w^1, \ldots w^m\}$ is the set $U$ of payoff vectors corresponding to *pure* strategies.

[2] Of course, for low discount factors, deterministic sequences cannot duplicate the effect of public randomization. If $\delta$ is near zero, the payoff vector for the sequence $\{u(r)\}$ is approximately $g(a(1))$, and so, quite apart from equilibrium considerations, many payoffs in $V^*$ are not feasible.

LEMMA 1. *If $\delta > 1 - 1/m$, then for any $v \in V$ there is a sequence $\{a(t)\}$ of pure strategies whose normalized payoffs are v.*

*Proof:* Let $v = \sum \lambda^k w^k$, where $0 \leqslant \lambda^k \leqslant 1$, and $\sum_{k=1}^{m} \lambda^k = 1$. We construct $\{a(t)\}$ as follows. Let $I^k(t)$ be an index variable, which is 1 if $a(t) = a^k$ and 0 otherwise. Set $N^k(1) = 0$ for all $k$, and let

$$N^k(t) = \sum_{\tau=1}^{t-1} (1-\delta)\,\delta^{\tau-1} I^k(t) \text{ for } t > 1,$$

and

$$C(t) = \{k \mid \lambda^k - N^k(t) > \delta^{t-1}(1-\delta)\}.$$

Now define

$$k^*(t) = \arg\max_{k \in C(t)} \{\lambda^k - N^k(t)\},[3]$$

and set $a(t) = a^{k^*(t)}$ This defines an algorithm for computing $a(t)$. One can verify that the sequence $\{a(t)\}$ has discounted average payoffs v. (See [4] for the details.)

LEMMA 2. *For any $\varepsilon > 0$ there exists $\underline{\delta} < 1$ such that for all $\delta \geqslant \underline{\delta}$ and every $v \in V^*$ with $v_i \geqslant \varepsilon$ for all i there is a deterministic sequence of pure strategies whose discounted average payoffs are v, and whose continuation payoffs at each time t are within $\varepsilon$ of v.*

*Remark.* To prove that public randomization is inessential for the Folk Theorem with observable mixed strategies, the uniformity of $\underline{\delta}$ (the fact that a single $\underline{\delta}$ applies to all v with $v_i \geqslant \varepsilon$) is not needed. However, this stronger property is needed to show that public randomization is inessential when mixed strategies are unobservable and is also needed in our paper [5].

*Proof.* (A) First we show that for each $v \in V^*$ there is a neighborhood over which the conclusion of the lemma holds. Given v and $\varepsilon$, let $\varepsilon' = \varepsilon/4$, and let $B(v; \varepsilon') = \{v' \in V^* \mid \|v' - v\| < \varepsilon'\}$ be the ball of radius $\varepsilon'$ centered at v. Let $Z$ be a polygon with vertices $\{z^l\}$ such that

(i) each $z^l$ is within $2\varepsilon'$ of v,

(ii) every $v' \in B(v; \varepsilon')$ can be expressed as a convex combination of the $\{z^l\}$, and

(iii) each $z^l$ can be expressed as $\sum_{k=1}^{m} \lambda^k(l)\, w^k$, where each weight $\lambda^k(l)$ is a rational number between zero and one, and the weights sum to 1.

[3] If there is a tie, make a deterministic selection.

Because the $\lambda^k(l)$'s are rational, we can find integers $d$ and $\{r^k(l)\}_{k=1}^m$ such that for all $l$ and $k$, $\lambda^k(l) = r^k(l)/d$. Let "cycle $l$" be the d-period sequence of pure strategies in which $a^1$ is played for the first $r^1(l)$ periods; $a^2$ is played for the next $r^2(l)$ periods; and so on for all $k$ between 3 and $m$. (Recall that $w^k = g(a^k)$). Let $z^l(\delta)$ be the discounted average payoffs corresponding to cycle $l$. If we set

$$R^k(l) = \sum_{s=1}^{k} r^s(l), \qquad \text{with} \quad R^0(l) = 0,$$

then

$$z^l(\delta) = \sum_{k=1}^{m} \sum_{s=R^{k-1}(l)}^{R^k(l)-1} (1-\delta)\,\delta^s w^k/(1-\delta^d).$$

We now apply the algorithm of Lemma 1 to generate each $v' \in B(v; \varepsilon')$ by a deterministic sequence of the $z^l(\delta)$'s for $\delta > 1 - 1/m$. Earlier, when the payoffs $w^k$ were called for in a given period $t$, we set $a(t) = a^k$. In our current application, we replace the $w^k$'s with the $z^l(\delta)$'s. Moreover, when the algorithm calls for payoffs $z^l(\delta)$, we assign cycle $l$ as the actions for the next $d$ periods. The Lemma 1 algorithm so modified guarantees that we can generate each of the payoff vectors in $B(v; \varepsilon')$ by a deterministic sequence of these cycles. Because each cycle is of length $d$ and each $z^l(\delta)$ gives each player a payoff within $3\varepsilon'$ of $v'$, the continuation payoffs starting at any time $t$ are within $3\varepsilon'$ of $v'$ if $\delta$ is chosen large enough that $(1-\delta^d)\max_a \| g(a) - v' \| < 3\varepsilon'$. We conclude that for all $v \in V^*$ and all $\varepsilon > 0$ there is a $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$ and all $v' \in B(v; \varepsilon/4)$ there is a deterministic sequence whose payoffs are $v'$ and whose continuation payoffs at each date are within $\varepsilon$ of $v$ and $v'$.

(B) Next consider the set $Q = \{v \in V^* \mid v_i > \varepsilon \text{ for all } i\}$. The collection $\mathbb{B} = \bigcup_{v \in Q} B(v; \varepsilon/4)$ is an open cover of $Q$, and $Q$ is compact, so the collection $\mathbb{B}$ contains a finite subcover. That is, there is a finite collection $F$ of the $\{B(v; \varepsilon/4)\}$ whose union contains $Q$. If we choose $\underline{\delta}^*$ to be the maximum of the associated $\underline{\delta}$'s, we conclude that if $\delta > \underline{\delta}^*$ then for all $v \in Q$ there is a deterministic sequence with the properties asserted by the lemma.

Q.E.D.

To summarize, the algorithm of lemma 1 shows how to attain any $v \in V$ by a deterministic sequence of $w^k$'s. Lemma 2 replaces the $w^k$'s with vectors $z^l(\delta)$ that are close to v and can be attained through a finite cycle of $w^k$'s. Then, to obtain $v \in V^*$ through a deterministic sequence, we (i) apply the Lemma 1 algorithm using the $z^l(\delta)$'s instead of the $w^k$'s; and (ii) whenever the algorithm calls for $z^l(\delta)$, replace it with the corresponding d-period cycle.

## 4. The Folk Theorem without Public Randomization

Now we can show that public randomization is not required to obtain the Folk Theorem in discounted repeated games. We begin with the simpler case in which players' mixed strategies are observable (or, equivalently, in which we restrict attention to payoffs that exceed the pure-strategy minimax levels.) The strategies we use are very similar to those in our [3] proof for games where public randomization is available (Theorem 2, pp. 544–545), which have the following form: To obtain payoff vector $v \in V^*$, players begin in "Phase (A)," where they use a public randomization that yields expected payoff v in every period so long as no player deviates. To deter deviations, each player is threatened with a "punishment equilibrium." The punishment equilibrium for player *i* has two phases. Players begin in phase $(B^i)$, in which *i's* opponents play the minimax strategies $m^i_{-i}$ for a specified number of periods $N^i$. If there are no deviations, play then switches to a "reward phase" (C'), in which the payoff vector is *v'(i),* where *v'(i)* is chosen so that each player prefers not to deviate in Phases $(B^i)$ or $(C^i)$ given that doing so would cause him to be punished in turn. Like v, the payoffs o'(i) are obtained by public randomization.

To adapt this construction to games without public randomization, we replace the randomizations yielding v and *v'(i)* with deterministic sequences that generate the same payoffs and whose continuation payoffs are close to the target levels.

**Proposition** 1. *Consider an n-player game in which players' choices of mixed strategies are observable and public randomization is not available. Assume that the dimension of V* equals n, i.e., that the interior of V* relative to* $\mathbb{R}^n$ *is non-empty. Then for any* $v = (v_1, \ldots, v_n) \in V^*$ *there is a* $\underline{\delta} < 1$ *such that for all* $\delta \in (\underline{\delta}, 1)$ *there is a subgame-perfect equilibrium of the infinitely repeated game with discount factor* $\delta$ *in which the discounted average payoffs are v.*

*Proof* Choose v' in the interior of *V** such that $v'_i < v_i$ for all *i*. Take $\rho > 0$ such that, for all players *i*, the vector $v'(i) = (v'_1 + \rho, \ldots, v'_{i-1} + \rho, v'_i, v'_{i+1} + \rho, \ldots, v'_n + \rho)$ is in *V**. Set v'(0) = v. Let $w^j_i = g_i(m^j)$ be player *i's* period payoff when j is being punished with the strategies $m^j$. Choose $\varepsilon > 0$ such that, for all *i* and *j*, $\varepsilon < v'_i$ and

$$-w^j_i < \frac{v'_i - \varepsilon}{v'_i}(\rho - w^j_i). \tag{1}$$

From Lemma 2, for all $\delta$ greater than some $\delta_\varepsilon$ and each $i = 0, \ldots, n$ there is a deterministic sequence $\{a^i(t, S)\}$ whose average payoffs are *v'(i)* and whose continuation payoffs at each date are within $\varepsilon$ of $v'(i)$. Choose $\underline{\delta} > \delta_\varepsilon$ such that for all $\delta > \underline{\delta}$ there exists an integer N(6) for which, for all $i$ and $j$,

$$(1-\delta)\,\bar{v}_i + \delta^{N(\delta)+1} v'_i < v'_i - \varepsilon, \tag{2}$$

$$(1-\delta)\,\bar{v}_i + \delta^{N(\delta)+1} v'_i < (1-\delta^{N(\delta)})\, w_i^j + \delta^{N(\delta)}(v'_i + \rho), \tag{3}$$

and

$$(1-\delta)\,\bar{v}_i + \delta^{N(\delta)+1} v'_i < (1-\delta)\, w_i^j + \delta(v'_i + \rho). \tag{4}$$

If there are several integers satisfying (2)-(4) let N(6) be the smallest one. Formula (1) ensures that $\underline{\delta}$ exists. (If (1) holds, we can find x $\in$ (0, 1) such that $xv'_i < v'_i - \varepsilon$ and $xv'_i < (1-x)\, w_i^j + x(v'_i + \rho)$ and then we can choose $\underline{\delta}$ such that $\underline{\delta} \approx 1$ and N(6) such that $\delta^{N(\delta)} \approx$ x.)

Now consider the following candidate equilibrium strategy for each player $i$:

(A) Begin by playing the sequence $\{a_i^0(t, \delta)\}$, and continue to do so as long as $\{a^0(t, \mathrm{S})\}$ was played the previous period or at least two players deviated that period.

If a single player $j$ deviates from (A), then

($\mathbf{B}^j$) Play $m_i^j$ for N(6) periods,

and then

($\mathbf{C}^j$) Play $\{a_i^j(t, \mathrm{S})\}$ thereafter.

If player $k$ deviates in phase ($\mathbf{B}^j$) or ($\mathbf{C}^j$), then begin phase ($\mathbf{B}^k$). (As in phase (A), ignore simultaneous deviations by two or more players).

To confirm that these are equilibrium strategies it suffices to check that no player can gain from deviating once and then conforming.

(i) If player $i$ deviates at any date $t$ in phase (A) and then conforms, he receives at most

$$\bar{v}_i(1-\delta) + \delta^{N(\delta)+1} v'_i. \tag{5}$$

Condition (2) ensures that this is less than $v'_i - \varepsilon$, which is a lower bound for player i's continuation payoff to conforming from date $t$ on.

(ii) If player $i$ conforms in phase ($\mathbf{B}^i$), when he is being minimaxed, he obtains at least

$$p_i \equiv \delta^{N(\delta)} v'_i > 0.$$

If he deviates once and then conforms, he receives at most zero in the period he deviates, and $\delta p_i$ thereafter, which is less than $p_i$.

(iii) If player $i$ deviates in the tth period of phase $(B^j)$, $j \neq i$, he obtains at most the value displayed in (5). If he conforms, he obtains at least

$$(1-\delta^{N(\delta)-t+1})\, w_i^j + \delta^{N(\delta)+t+1}(v_i' + \rho). \tag{6}$$

If $t = 1$, then (3) implies that (6) exceeds (5). If $t = N(\delta)$, this inequality follows from (4); intermediate cases follow from (3) and (4) combined.

(iv) Finally, in phase $(C^k)$, player *i's* continuation payoff if he conforms is at least $v_i' - \varepsilon$, and so condition (2) ensures that deviation is not profitable.

We conclude that no player can gain by deviating at any date of any phase. Q.E.D.

***Remark.*** Note that we have taken the length of the punishment phase, $N(\delta)$, to be the same for all players.

The case where only players' realized actions (and not the randomizations themselves) are observable presents an additional complication. If, to minimax player $i$, player $j$ uses a mixed strategy, he must be indifferent among the various actions over which he randomizes. Our proof in [3] ensured this indifference by making j's continuation payoff after the punishment phase contingent on his actions during the phase. It is important here that precisely specified values for the continuation payoffs be attainable; it would not suffice merely two approximate them. We should point out that [3] was misleading in its assertion that the Folk Theorem holds approximately when public randomization is not allowed. If certain continuation payoffs could only be approximated, and not attained exactly, our methods would yield an approximate Folk Theorem for games with *observable* minimax strategies, but they would not provide even an approximate Folk Theorem for the case where the minimax strategies are mixed and mixed strategies cannot be directly observed.

The proof that the Folk Theorem does indeed obtain with unobservable minimax strategies and no public randomization relies on the fact that, in Lemma 2, $\underline{\delta}$ can be chosen to hold uniformly over all payoff vectors that are bounded away from the coordinate axes. This uniformity is required because the payoffs we specify in the reward phases will now depend on the discount factor. Without uniformity we would run into the difficulty that if $\underline{\delta}$ were chosen large enough to invoke Lemma 2 for particular continuation payoffs, the continuation payoffs required to ensure indifference would themselves change, necessitating the choice of a new $\underline{\delta}$, and so on.

PROPOSITION 2. ***Consider an n-player game in which public randomization is not available and only the players' choices of action are observable.***

*Assume that the dimension of $V^*$ equals $n$.*[4] *Then for any $v \in V^*$ there is $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a subgame perfect equilibrium of the infinitely repeated game with discount factor $\delta$ in which the discounted average payoffs are $v$.*

*Proof:* The repeated game strategies that we use are very similar to those of Proposition 1. Once again, set $v'(0) = v$ and choose $v'$ in the interior of $V^*$ so that $v'_i < v_i$ for all $i$. Take $\rho > 0$ so that for all players $i$, the vector $v'(i) = (v'_1 + \rho, \ldots, v'_{i-1} + \rho, v'_i, v'_{i+1} + \rho, \ldots, v'_n + \rho)$ is in $V^*$. Choose $\varepsilon$ satisfying (2) and $0 < \varepsilon < \min_i v'_i$, and consider $\delta_\varepsilon$ sufficiently large that, for all $\delta > \delta_\varepsilon$, there is a deterministic sequence $\{a^i(t, \delta)\}$ whose discounted average payoffs are $v'(i)$ and whose continuation payoffs are within $\varepsilon$ of $v'(i)$.

Consider the following repeated game strategy for player $i$:

(A) Begin by playing the sequence $\{a^0_i(t, \delta)\}$ and continue to do so as long as $\{a^0(t, \delta)\}$ was played the previous period or at least two players deviated that period.

If player $j$ unilaterally deviates from (A), then

($B^j$) Play $m^j_i$ for $N(\delta)$ periods, where $N(\delta)$ is the smallest integer that satisfies (2)-(4). If player $k$ unilaterally chooses an action outside the support of $m^j_k$, go to phase $B^k$; ignore simultaneous deviations.

At the end of Phase (B), play switches to Phase ($C^j$), which requires some preliminary notation.

Define $r^j_i$ to be player $i$'s realized discounted average payoff during phase ($B^j$). (Note that if players use mixed strategies, then $r^j_i$ is a random variable.) That is, if $a(1), a(2), \ldots, a(N(\delta))$ are the realized actions in Phase ($B^j$), then $r^j_i = (1 - \delta) \sum_{\tau=1}^{N(\delta)} \delta^{\tau-1} g_i(a(\tau))/(1 - \delta^{N(\delta)})$. Set

$$z^j_i = \begin{cases} r^j_i(1 - \delta^{N(\delta)})/\delta^{N(\delta)}, & i \neq j \\ 0, & i = j \end{cases} \tag{7}$$

[For readers familiar with our [3], we point out that this construction is slightly different in working with the realized average payoff instead of the excess between that average and what player $i$ gets from his least preferred strategy in the support of $m^j_i$.]

Let $\{a(t, \delta, \{z^j_i\})\}$ be a deterministic sequence that results in the payoffs

$$(v'_1 + \rho - z^j_1, \ldots, v'_{j-1} + \rho - z^j_{j-1}, v'_j, v'_{j+1} + \rho - z^j_{j+1}, \ldots, v'_n + \rho - z^j_n), \tag{8}$$

and all of whose continuation payoffs are within $\varepsilon$ of (8). We will show below that such a sequence exists for $\varepsilon$ close enough to 0 and $\delta$ near enough 1.

[4] In [5] we show that this hypothesis can be dropped when $n = 2$.

Now we define strategies in Phase $(C^j)$:

$(C^j)$ Play $a_i(t, \delta, \{z_i^j\})$ unless player $k$ unilaterally deviates, in which case go to $(B^k)$.

The phase $(C^j)$ strategies are constructed so that each player $i \neq j$ is indifferent among all actions during punishment phase $(B^j)$ (if he conforms in Phase $(C^j)$: Regardless of player $i$'s actions in $(B^j)$, his continuation payoff from the beginning of Phase $(B^j)$ is

$$(1\text{-}6)\quad \sum_{\tau=1}^{N(\delta)} \delta^{\tau-1} g_i(a(\tau)) + \delta^{N(\delta)}(v_i + \rho + z_i^j) = \delta^{N(\delta)}(v_i + \rho).$$

The arguments that player $i$ will not deviate in phase (A) or $(C^j)$ are exactly the same as those in the proof of Proposition 1. It remains to show that, for $\varepsilon$ near enough to 0 and $\delta$ near enough to 1, there exists a deterministic sequence generating the payoffs (8) with continuation payoffs uniformly close to (8). Consider a sequence $\{(\varepsilon_n, \delta_n)\}$, where $\varepsilon_n$ tends to 0 and $\delta_n$ to 1. Formula (2) can be rewritten as

$$\delta^{N(\delta)+1} < (v_1' - \varepsilon - (1-\delta)\,\bar{v}_i)/v_i',$$

and the right-hand side tends to 1 as $\varepsilon \to 0$ and $\delta \to 1$. Since N(6) is the smallest integer satisfying (2)–(4), this implies that $\delta_n^{N(\delta_n)} \approx 1$ for $n$ sufficiently large. Hence, from (7), $z_i^j \approx 0$ and $\rho - z_i^j > 0$. Moreover, for large $n$ the payoffs (8) are in the interior of $V^*$ and bounded away by at least $\varepsilon_n$ from the axes. Lemma 2 then implies that for $n$ sufficiently large there exists $\underline{\delta}$ near enough 1 (and greater than 6,) such that, for $\delta > \underline{\delta}$, there are sequences generating (8) that have continuation payoffs within $\varepsilon_n$ of (8).

Q.E.D.

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