



ELSEVIER

European Economic Review 44 (2000) 667–681

www.elsevier.com/locate/econbase

EUROPEAN  
ECONOMIC  
REVIEW

Alfred Marshall Lecture

# Auctions, development, and privatization: Efficient auctions with liquidity-constrained buyers

Eric S. Maskin\*

*Department of Economics, Harvard University, Littauer Center, Cambridge, MA 02138, USA  
MIT, Cambridge, MA 02139, USA*

---

## Abstract

We exhibit an efficient auction – an auction for which the winner is the buyer with the highest valuation – subject to the constraint that buyers may be budget- or liquidity-constrained. © 2000 Elsevier Science B.V. All rights reserved.

*JEL classification:* D44; D82

*Keywords:* Efficient auctions; Liquidity constraints

---

The three subjects in the title of this lecture all have vast literatures associated with them. My concern is mainly with their interconnection. Development – by which I mean enhancing an economy’s productivity or its distribution of goods and services – can be thought of as the goal; whereas privatization – the transfer of productive assets from public to private hands – can be thought of as the blueprint or architecture for achieving that goal. Finally, auctions – mechanisms for selling off the productive assets – provide the engineering for executing the blueprint.

Privatization is an instrument that, of course, has been widely used in recent years – from Eastern Europe to China, from sub-Saharan Africa to Russia. There are at least five reasons why privatization has been widely considered an

---

\*E-mail address: emaskin@mit.edu and emaskin@harvard.edu

effective way of encouraging development. First, it promotes *efficiency*, that is, it gets capital into the hands of those who can use it most productively. Second, it promotes *competition*: it helps restrain prices and keeps entrepreneurs on their toes by ensuring that they have viable rivals. Third, it can be an effective method of generating *revenue* for public projects. Fourth, because different assets have different distributions of returns, it is a way of *allocating risks* across members of the economy. Fifth, it is a tool for *redistributing income*. Of course, some may view privatization as a goal in its own right. That is, one may believe that it is politically or morally preferable to have capital under private rather than state control. But this last objective will be attained automatically if privatization occurs for any of the other reasons.

In this lecture, I will particularly stress the rationale of *efficiency*. This is not to say that the other reasons are unimportant. But almost invariably when privatization is contemplated, efficiency is invoked as a leading justification. This has been true not only in developing economies, but in major industrial nations such as the U.K. and the U.S. For example, in the recent American privatization of radio frequency bands for telecommunications, the Federal Communications Commission was mandated to award licenses to those who could make the best economic use of them.

Furthermore, efficiency is, to a considerable extent, not at odds with other rationales. Indeed, it often complements them well. Take raising revenue, for example. Devising an auction to maximize the revenue from the sale of capital is not the same thing as devising one for maximizing efficiency. However, the two objectives are closely related: a buyer who can make the best use of an asset is also quite likely to be willing to pay the most for it.

There is, however, at least one important qualification to this logic. Specifically, the entrepreneurs with the best plans for using the capital assets may be liquidity-constrained. That is, they may not be able to afford what the assets are worth to them. This qualification is particularly relevant in developing economies, where credit markets that could provide the needed liquidity may not function well. As we shall see, liquidity constraints can present a major obstacle to the efficient allocation of capital.

This motivates the major theme I wish to discuss: how to go about designing an auction for the efficient allocation of capital when entrepreneur-buyers may be significantly budget-constrained.

## 1. Model

The model that we will study consists of one (indivisible) unit of a capital asset to be privatized and  $n$  potential (risk neutral) buyers, indexed by  $i = 1, \dots, n$ . Let us imagine that each buyer proposes to use the asset to produce some product for sale. Buyer  $i$  has a valuation  $v_i$  for the asset (corresponding to the marginal

profit from the sale of the product), where  $v_i$  is drawn from a probability distribution represented by c.d.f.  $F_i(v_i)$  with support  $[0, \bar{v}_i]$ . We shall assume that the  $v_i$ 's are private information, that the  $v_i$ 's are drawn independently, and the  $F_i$ 's are common knowledge. If buyer  $i$  makes a monetary payment  $R_i$  and has probability of winning  $p_i$ , his net payoff is

$$p_i v_i - R_i.$$

An *auction* is a game (mechanism) that determines (1) who wins (which buyer is allocated the asset), and (2) how much each buyer pays (this allows for the possibility that the winner may not be the only buyer who makes a payment). Two familiar examples of auctions include the *high-bid auction*, in which each buyer  $i$  makes a bid  $b_i$ , the winner is the buyer with the highest bid, and the winner pays his bid; and the *second-price auction*, wherein the rules are the same as for the high-bid auction except that the winner pays the second-highest bid rather than his own bid.

## 2. Efficiency

An auction is *efficient* if, in equilibrium, the winner is the buyer with the highest valuation, i.e., if buyer  $i$  wins then

$$v_i \geq v_j \quad \text{for all } j \neq i. \quad (1)$$

This criterion of efficiency implicitly assumes that if the asset is awarded to buyer  $i$ , then the social benefit from the asset equals  $v_i$  (buyer  $i$ 's private benefit). For such a coincidence of private and social benefits, it is important that there be no externalities (although, as we shall see below, our analysis can be generalized to the case in which externalities are present) and that the winner is situated in a competitive market, that is, there are many close substitutes for the product it sells (if this were not the case, then efficiency might call for regulation of the winner).

A reasonable question to ask is why we should worry about an auction's efficiency. After all, if the initial assignment of the asset were incorrect, we might expect trade afterwards to correct the problem. This might indeed be a reasonable expectation if the number of buyers were large. However, with small numbers, retrade will not in general work. To see this imagine that there were just two buyers. Suppose that the asset were initially allocated to buyer 1, for whom  $v_1 = 1$ . Let us imagine that, from buyer 1's perspective, buyer 2's valuation is drawn from a uniform distribution on the interval  $[0, 2]$ . Then if buyer 1 attempts to resell the asset, he will offer it at price  $p$  to maximize his expected net gain

$$\frac{1}{2}(2 - p)(p - 1).$$

That is, he will set  $p = \frac{3}{2}$ . But if  $v_2$  turns out to be between 1 and  $\frac{3}{2}$ , the asset will not be sold (buyer 2 will refuse the price), even though it would be more efficient for buyer 2 to possess the asset (by contrast, with a large number of alternative buyers, buyer 1 could be nearly certain of getting a taker at his profit-maximizing price).

### 3. The second-price auction

As we will see, none of the standard auctions – including the high-bid and second-price auctions – are efficient in the sense of (1) once we introduce liquidity constraints for buyers. Let us note first, however, that if buyers are not limited in the prices they can pay, the second-price auction is efficient.

*Proposition 0* (Vickrey, 1961). *In the second-price auction, it is optimal for each buyer  $i$  to bid his reservation price  $b_i = v_i$ .<sup>1</sup>*

*Proof.* Notice first that there is no point in buyer  $i$  bidding less than  $v_i$ ; this would just reduce his chance of winning, but would not reduce his payment (because he does not pay his bid anyway). By bidding more than  $v_i$ , he can increase his chance of winning. But the only additional cases in which he would now win are those in which some other buyer  $j$  bids  $b_j > v_i$ . And observe that this means that buyer  $i$  pays (at least)  $b_j$ , which is more than the asset is worth to him. So  $b_i = v_i$  is indeed optimal.  $\square$

*Corollary.* The second-price auction is efficient.

*Proof.* Because buyers bid their valuation in equilibrium, the buyer with the highest valuation wins.  $\square$

We can readily modify the second-price auction to accommodate externalities. Specifically, suppose that buyer  $i$ 's project (his use of the asset) would impose an external effect  $e_i$  on the 'community'. Then efficiency implies that the winning buyer  $i$  be such that

$$v_i + e_i \geq v_j + e_j \quad \text{for all } j \neq i. \quad (2)$$

Consider the following auction:

- (a) each buyer  $i$  bids  $b_i$  and the community enters a vector of bids  $(d_1, \dots, d_n)$ ;
- (b) the winning buyer is the buyer  $i$  for whom  $b_i + d_i \geq b_j + d_j$  for all  $j \neq i$ ;

---

<sup>1</sup> Formally speaking, bidding  $b_i = v_i$  is a dominant strategy.

(c) the winner pays  $\max_{j \neq i} (b_j + d_j) - d_i$ ;

(d) the community pays  $b_k - b_i$ , where  $k = \arg \max_j b_j$ .

We claim that it is optimal for buyer  $i$  to set  $b_i = v_i$ . Notice from (b) that, if he does so, he wins if and only if

$$v_i \geq \max_{j \neq i} (b_j + d_j) - d_i. \quad (3)$$

But from (c) the right-hand side of (3) is what buyer  $i$  pays if he wins. Thus by bidding truthfully, he ensures that he wins if and only if his net payoff is positive, which, given that his payment is independent of his bid, is the best he can do. We also claim that the community will set  $(d_1, \dots, d_n) = (e_1, \dots, e_n)$ . Observe from (b) that, if they do so, the winner will be buyer  $i$  such that

$$b_i + e_i \geq b_j + e_j \quad \text{for all } j \neq i, \quad (4)$$

and from (d), the community's net payoff will be

$$e_i - b_k + b_i. \quad (5)$$

If instead the community bids so that buyer  $j$  wins, its payoff will be

$$e_j - b_k + b_j. \quad (6)$$

But from (4), (5) is not less than (6). Hence,  $(d_1, \dots, d_n) = (e_1, \dots, e_n)$  is optimal. Given that all participants bid truthfully, (4) ensures that the winning buyer is buyer  $i$  for whom

$$v_i + e_i \geq v_j + e_j \quad \text{for all } j \neq i,$$

which implies efficiency.

#### 4. Liquidity constraints

Suppose each buyer  $i$  has a budget limitation  $w_i$ . This has the implication that a buyer cannot make a bid for which he may have to pay more than  $w_i$ . If  $w_i > v_i$ , this limitation has no effect on bidding in the second-price auction, since by bidding optimally, the buyer ensures that he never pays more than  $v_i$ . If, however,  $w_i < v_i$ , then bidder  $i$  cannot bid  $b_i = v_i$ , since then he may have to pay more than  $w_i$ . Indeed, he cannot bid more than  $w_i$ , and so will optimally make exactly this bid.

But this constraint can readily compromise the efficiency of the second-price auction. Suppose, for example, that

$$w_1 < w_2 < v_2 < v_1.$$

Then, each buyer  $i$  bids  $w_i$ , and so buyer 2 wins. But notice that the efficient outcome would have buyer 1 win.

There are several potential ways around this difficulty. First, the auctioneer might allow the winning buyer to pay for the asset out of future earnings (presumably,  $v_i$  represents anticipated profit). However, such an arrangement would require earnings to be observable ex post; otherwise the buyer could simply claim that he did not have any. Moreover, even if earnings were observable ex post, actually collecting payments after the fact would require an enforcement mechanism – an institution that might be difficult to come by in a developing economy. Finally, if earnings were stochastic and the buyer could influence their distribution, a moral hazard problem might well arise if the buyer were supposed to pay some of these earnings back.

Another way to address the liquidity-constraint problem would be to give buyers monetary transfers to help finance their purchases. Of course, such a scheme might also encounter moral hazard difficulties; in particular, it could well encourage non-serious bidders to participate (in order to collect the transfer). It could also be expensive, particularly in developing countries where the deadweight loss associated with generating public funds (out of which the transfers are presumably paid) is typically quite high.

In view of the difficulties associated with transfer and ex post payment schemes – difficulties that seem particularly acute in developing economies – we will henceforth simply rule them out.

## 5. Constrained efficiency

Once buyers face budget constraints, *no* auction can, in general, be fully efficient, i.e., efficient in the sense of (1). (To see this, consider the case in which  $w_i = 0$  for all  $i$ . Then buyers' bids cannot be used at all to determine the allocation, and so full efficiency is obviously impossible.) The natural questions, therefore, are whether we can improve upon the degree of efficiency attained by the second-price auction and, if so, what the nature of a *constrained*-efficient auction is.

Let us restrict attention for the time being to the case of two buyers. In any auction, an equilibrium can be summarized by the functions  $\{(\pi_i(v_1, v_2), b_i^W(v_1, v_2), b_i^L(v_1, v_2))\}_{i=1,2}$ , where, given valuations  $(v_1, v_2)$ ,  $\pi_i(v_1, v_2)$  is the probability that buyer  $i$  wins,  $b_i^W(v_1, v_2)$  is the payment that buyer  $i$  makes if he wins, and  $b_i^L(v_1, v_2)$  is the payment he makes if he loses. The degree of efficiency of the auction can be measured by its expected surplus:

$$\int_0^{\bar{v}_2} \int_0^{\bar{v}_1} (\pi_1(v_1, v_2)v_1 + \pi_2(v_1, v_2)v_2) dF_1(v_1) dF_2(v_2). \quad (7)$$

Hence, we can think of the problem of finding constrained efficient auction as that of maximizing (7) subject to incentive-compatibility

$$v_1 \in \arg \max_{\hat{v}_1} \int [\pi_1(\hat{v}_1, v_2)(v_1 - b_1^W(\hat{v}_1, v_2)) + (1 - \pi_1(\hat{v}_1, v_2))(-b_1^L(\hat{v}_1, v_2))] dF_2(v_2) \tag{8a}$$

and

$$v_2 \in \arg \max_{\hat{v}_2} \int [\pi_2(v_1, \hat{v}_2)(v_2 - b_2^W(v_1, \hat{v}_2)) + (1 - \pi_2(v_1, \hat{v}_2))(-b_2^L(v_1, \hat{v}_2))] dF_1(v_1), \tag{8b}$$

to budget constraints

$$b_i^W(v_1, v_2) \leq w_i \quad \text{and} \quad b_i^L(v_1, v_2) \leq w_i, \quad i = 1, 2, \tag{9}$$

and to the no-transfer constraints

$$0 \leq b_i^W(v_1, v_2) \quad \text{and} \quad 0 \leq b_i^L(v_1, v_2), \quad i = 1, 2. \tag{10}$$

The integral in (8a) is buyer 1’s expected payoff if his valuation is  $v_1$  but he bids as though his valuation were  $\hat{v}_1$ . Condition (8a) is, therefore, the requirement that buyer 1 should maximize his expected payoff by bidding according to his actual valuation  $v_1$ . Condition (8b) is the corresponding requirement for buyer 2.

We first argue that, in seeking a constrained efficient auction, we can assume that there exist functions  $b_1(\cdot)$  and  $b_2(\cdot)$  such that

$$b_1^W(v_1, v_2) = b_1^L(v_1, v_2) = b_1(v_1) \quad \text{for all } (v_1, v_2) \tag{11a}$$

and

$$b_2^W(v_1, v_2) = b_2^L(v_1, v_2) = b_2(v_2) \quad \text{for all } (v_1, v_2). \tag{11b}$$

That is, in a constrained-efficient auction, a buyer’s payment depends only on his valuation and not on whether he wins or loses. To see this, consider an auction  $\{(\pi_i(v_1, v_2), b_i^W(v_1, v_2), b_i^L(v_1, v_2))\}_{i=1,2}$  satisfying (8a), (8b), (9), and (10). Let us argue from the perspective of buyer 1 (the argument is symmetric for buyer 2). For any  $v_1$ , define  $b_1(v_1)$  so that

$$b_1(v_1) = \int [\pi_1(v_1, v_2)b_1^W(v_1, v_2) + (1 - \pi_1(v_1, v_2))b_1^L(v_1, v_2)] dF_2(v_2). \tag{12}$$

Because  $b_1^W(v_1, v_2)$  and  $b_1^L(v_1, v_2)$  satisfy (9) and (10), (12) implies that  $b_1(v_1)$  does. Moreover, (12) implies that

$$v_1 \in \arg \max_{\hat{v}_1} \int \pi_1(\hat{v}_1, v_2)v_1 dF_2(v_2) - b_1(\hat{v}_1). \tag{13}$$

Thus  $b_1(v_1)$  satisfies all the constraints in the efficiency program. Now, note that if  $b_1^W(v_1, v_2) = w_1$  and  $b_1^L(v_1, v_2) < w_1$  for some  $(v_1, v_2)$ , then  $b_1(v_1)$  defined by

(12) satisfies  $b_1(v_1) < w_1$ . That is, when the budget constraint is binding for  $b_1^W(v_1, v_2)$ , it will typically be *relaxed* by replacing  $b_1^W(v_1, v_2)$  and  $b_1^L(v_1, v_2)$  with  $b_1(v_1)$ . Thus, a greater degree of efficiency becomes possible.

Put less formally, what matters in order to satisfy buyer 1’s budget constraint is that

$$\max\{b_1^W(v_1, v_2), b_1^L(v_1, v_2)\} \leq w_1. \tag{14}$$

In an efficient auction, buyer 1 should win with probability  $F_2(v_1)$ . Thus, buyer 1 chooses  $\hat{v}_1$  to solve

$$\max_{\hat{v}_1} [F_2(\hat{v}_1)v_1 - \bar{b}_1(\hat{v}_1)],$$

where  $\bar{b}_1(\hat{v}_1)$  is buyer 1’s expected payment corresponding to  $\hat{v}_1$ . In equilibrium,  $\hat{v}_1 = v_1$  and so

$$F'_2(v_1)v_1 - \bar{b}'_1(v_1) = 0,$$

i.e.,

$$\bar{b}_1(v_1) = \int_0^{v_1} xF'_2(x) dx. \tag{15}$$

Hence, from (12) and (15),

$$\int_0^{v_1} xF'_2(x) dx = \int [\pi_1(v_1, v_2)b_1^W(v_1, v_2) + (1 - \pi_1(v_1, v_2))b_1^L(v_1, v_2)] dF_2(v_2). \tag{16}$$

The problem with the second-price auction insofar as satisfying (14) is that  $b_1^L(v, v_2) \equiv 0$ , and so, to satisfy (16),  $b_1^W(v_1, v_2)$  must be correspondingly big, and so might violate (14). Given (16), the auction for which (14) is most readily satisfied is one in which  $b_1^W(v_1, v_2) = b_1^L(v_1, v_2)$ , which was what we claimed.

To express the next property of a constrained-efficient auction (i.e., of the solution to (7)–(10)), let us assume that  $w_1 = w_2 = w$  (if  $w_1 \neq w_2$ , the constrained efficient auctions could be a bit more complicated than what follows below; however, all our qualitative conclusions still follow). Let  $p$  and  $v^*$  be such that

$$(p + (1 - p)F_2(v^*))v^* - \int_0^{v^*} F_2(x) dx = w \tag{17a}$$

and

$$((1 - p) + pF_1(v^*))v^* - \int_0^{v^*} F_1(x) dx = w. \tag{17b}$$

We claim that, in a constrained-efficient auction,

$$\begin{aligned} & \text{Prob}\{\text{buyer 2 wins when his valuation is } v_2\} \\ &= \begin{cases} F_1(v_2) & \text{if } v_2 < v^*, \\ 1 - p + pF_1(v^*) & \text{if } v_2 \geq v^*, \end{cases} \end{aligned} \tag{18a}$$

and

$$\begin{aligned} & \text{Prob}\{\text{buyer 1 wins when his valuation is } v_1\} \\ &= \begin{cases} F_2(v_1) & \text{if } v_1 < v^*, \\ p + (1 - p)F_2(v^*) & \text{if } v_1 \geq v^*. \end{cases} \end{aligned} \tag{18b}$$

That is, as long as at least one of  $v_1$  and  $v_2$  is less than  $v^*$  (the lowest valuation for which the buyers’ budget constraints are binding), the winner is the buyer with the higher valuation, i.e., the good is allocated efficiently. However, if both buyers’ valuations exceed  $v^*$ , then the winner is chosen to be buyer 1 with probability  $p$  and buyer 2 with probability  $1 - p$ .

To understand why (18a) and (18b) hold, let us consider why we might conceivably want to violate them. Presumably, we would refrain from allocating the good to the more efficient buyer for low values of  $v_1$  and  $v_2$  only if, by so doing, we could relax the budget constraint for higher values, thereby making efficient allocation possible for those higher values. It turns out, however, that introducing inefficiencies for such purposes is not advantageous. This is easiest to see when the c.d.f.s  $F_1$  and  $F_2$  are concave (the conclusion is true more generally, however). For example, suppose that the good were allocated efficiently for all values of  $v_1$  less than or equal to  $v_1^0$ . From (15), if  $v_1 = v_1^0$ , buyer 1’s expected payment would have to be

$$\int_0^{v_1^0} xF_2'(x) dx. \tag{19}$$

But if this exceeded  $w$  it would be infeasible, and so we might enquire whether introducing inefficient allocation for  $v_1 < v_1^0$  could allow us to reduce buyer 1’s payment when  $v_1 = v_1^0$ , thereby permitting efficient allocation in a neighborhood of this valuation. Let us suppose, for instance, that for values of  $v_1$  and  $v_2$  less than  $v_1^0$ , the good is allocated with one-half probability to either buyer, but that we maintain efficient allocation when  $v = v_1^0$ . This means that if  $v_1 < v_1^0$ , buyer 1 would have probability  $\frac{1}{2}F_2(v_1^0)$  of winning. For this to be incentive-compatible, buyer 2 must pay  $b(v_1^0)$  such that

$$F_2(v_1^0)v_1^0 - b_1(v_1^0) = \frac{1}{2}v_1^0F_2(v_1^0) - c, \tag{20}$$

where  $c \geq 0$  is the payment corresponding to  $v_1 < v_1^0$ . But if  $F_2$  is concave, it is easy to show that  $b_1(v_1^0)$  in (20) is *greater than* (19). That is, introducing the inefficiency for  $v_1 < v_1^0$  does not relax the budget constraint for  $v_1 = v_1^0$  after all.

To summarize, we have

*Proposition 1.* If buyers are constrained by budget  $w$ , then in a constrained-efficient auction the amount buyer  $i$  pays ( $\int_0^{v_i} xF'(x) dx$  if  $v_i < v^*$  and  $w$  otherwise, where  $v^*$  is defined by (17a) and (17b)) is contingent only on his valuation and does not depend on whether he wins or loses, and allocation of the asset is efficient provided that at least one buyer's valuation is sufficiently low (i.e., less than  $v^*$ ), so that his budget constraint is not binding. If both buyers' valuations exceed  $v^*$ , then buyer 1 wins with probability  $p$  and buyer 2 with probability  $1 - p$  ( $p$  and  $1 - p$  defined by (17a) and (17b)).

### 6. The all-pay auction

Proposition 1 describes the properties of a constrained-efficient auction, but does not tell us what auction rules would ensure those properties. That is, it does not address the implementation question. To answer this question, let us begin with the case in which buyers are ex ante symmetric, i.e.,  $w_1 = w_2 = w$  and  $v_1$  and  $v_2$  are drawn from the same c.d.f.  $F(\cdot)$  (We shall, for convenience, always maintain the assumption that  $w_1 = w_2 = w$ .) In this case, it is easy to see that the so-called *all-pay auction* is constrained-efficient. This is the auction in which buyers make bids (ties are resolved by coin flips), the high bidder wins, and buyers pay their bids whether they win or lose. Constrained efficiency follows from the fact that, given the symmetry, the high bidder is the buyer with the highest valuation (assuming that the budget constraint is not binding for both buyers). For similar reasons, Che and Gale (1996) and Laffont and Robert (1996) concluded that, when buyers are budget constrained, variants of the all-pay auction maximize revenue generation and, in particular, generate strictly more revenue than the high-bid or second-price auctions.

Let us compare the all-pay auction to the second-price auction for the case of two buyers, when  $v_1$  and  $v_2$  are drawn uniformly from  $[0, 1]$  and  $w = \frac{1}{4}$ . In this case a fully efficient allocation (which would be attainable only if we removed the budget constraints) would lead to expected surplus  $2 \int_{v_1=0}^1 \int_{v_2=v_1}^1 v_2 dv_2 dv_1 = \frac{2}{3} = \frac{512}{768}$ . In the second-price auction, inefficiency occurs only when  $v_i \geq \frac{1}{4}$  for  $i = 1, 2$ : in that case both buyers bid  $\frac{1}{4}$ , and so the asset is allocated randomly. Hence expected surplus is

$$2 \int_{v_2=0}^{1/4} \int_{v_1=v_2}^1 v_1 dv_1 dv_2 + \int_{v_2=1/4}^1 \int_{v_1=1/4}^1 v_1 dv_1 dv_2 = \frac{458}{768}.$$

In the all-pay auction, inefficiency occurs only when  $v_1 \geq v^*$  for  $i = 1, 2$ , where

$$\left(\frac{1}{2} + \frac{1}{2}v^*\right)v^* - \frac{1}{4} = \frac{1}{2}(v^*)^2. \tag{21}$$

To understand (21), note that  $v^*$  is the valuation at which the budget constraint is just binding. Thus a  $v^*$ -buyer pays  $\frac{1}{4}$  and wins with probability  $v^* + \frac{1}{2}(1 - v^*)$ , which accounts for the left-hand side of (21). For this to be incentive-compatible, a  $v^*$ -buyer must not gain from behaving as though his valuation were slightly less than  $v^*$ ; if he did behave in this way, he would win with probability (slightly less than)  $v^*$  and from (19), would pay (slightly less than)  $\int_0^{v^*} x dx = \frac{1}{2}(v^*)^2$ , accounting for the right-hand side of (21). Hence,  $v^* = \frac{1}{2}$ , and so expected surplus is

$$2 \int_{v_2=0}^{1/2} \int_{v_1=v_2}^1 v_1 dv_1 dv_2 + \int_{v_2=1/2}^1 \int_{v_1=1/2}^1 v_1 dv_1 dv_2 = \frac{496}{768}.$$

### 7. Asymmetries

Unfortunately, buyers must be ex ante symmetric for the all-pay auction to be efficient. To see this, consider the case of two buyers in which  $v_1$  is distributed uniformly on  $[0, 1]$  and  $v_2$  is distributed uniformly on  $[0, 10]$ . Suppose that there are no budget constraints. It is not hard to see that in the all-pay auction buyer  $i$ 's equilibrium bid  $b_i^{ap}(v_i)$  as a function of his valuation is strictly increasing (if, say,  $b_2^{ap}(\cdot)$  were not strictly increasing, this would mean that  $b_2^{ap}(v_2) = b$  for all  $v_2$  in some interval; but then buyer 1 would never bid just below  $b$  since he could discontinuously raise his probability of winning by bidding slightly more than  $b$ ; but this in turn means that buyer 2 could lower his bid from  $b$  without reducing his probability of winning, a contradiction). Furthermore, we must have

$$b_1^{ap}(1) = b_2^{ap}(10). \tag{22}$$

(If, to the contrary, we had  $b_2^{ap}(10) > b_1^{ap}(1)$ , then buyer 2 could reduce his bid by  $\epsilon$  so that

$$b_2^{ap}(10) - \epsilon > b_1^{ap}(1)$$

and still win with probability 1, a contradiction.) But from (22) and the fact that  $b_2^{ap}(\cdot)$  is strictly increasing,

$$b_1^{ap}(1) > b_2^{ap}(10 - \alpha) \quad \text{for } \alpha > 0,$$

which violates efficiency (since  $1 < 10 - \alpha$  for  $\alpha$  small).

Thus, in summary, the all-pay auction (a) enhances efficiency relative to the second-price auction by gearing buyers' payments to their *expected* benefit from participating (so that their budget constraints are less likely to be violated), but (b) is nevertheless inefficient when buyers are asymmetric. The natural question, then, is whether there is a way to correct property (b) with sacrificing (a)? Happily, the answer is yes, at least in the case of three or more buyers.

**8. A constrained-efficient auction**

Let us stick to the case of 3 buyers. Suppose that  $w_1 = w_2 = w_3 = w$ . Then, by analogy with the two-buyer analysis above, a constrained-efficient auction has the properties that (i) if buyer 1’s valuation  $v_1$  is less than  $v^*$  (the value of  $v^*$  is given in the appendix), then he pays

$$\int_0^{v_1} x(F_2(x)F'_3(x) + F'_2(x)F_3(x)) dx \tag{23}$$

whether he wins or loses, and he wins provided that he has the highest valuation; and (ii) if buyer 1’s valuation  $v_1$  is greater than  $v^*$ , then he pays  $w$  whether he wins or loses, and he wins with probability  $p_1/(p_1 + p_j)$  if buyer  $j \neq 1$  is the only other buyer whose valuation exceeds  $v^*$ , with probability  $p_1$  if both other buyers’ valuations exceed  $v^*$  (the values of  $p_1, p_2,$  and  $p_3$  are provided in the appendix), and with probability 1 if both other buyers’ valuations are less than  $v^*$ ; (c) buyers 2 and 3 are treated completely analogously.

To understand where (23) comes from, notice that if buyer 1 wins with probability  $F_2(v_1)F_3(v_1)$  (the probability that his valuation is highest) then, for all  $v_1$ , his bid  $b_1(v_1)$  must satisfy

$$v_1 = \arg \max_{\hat{v}_1} [F_2(\hat{v}_1)F_3(\hat{v}_1)v_1 - b_1(\hat{v}_1)].$$

Hence,

$$\frac{d}{dv_1} (v_1 F_2(v_1) F_3(v_1)) = b'_1(v_1) \quad \text{for all } v_1 \tag{24}$$

and integrating (24) we obtain (23).

In an attempt to construct an auction with properties (i) and (ii), let us, as part of the auction, have buyer 2 report his own valuation and his assessments of buyer 1’s and 3’s probability distributions; call these reports  $\hat{v}_2, \hat{F}_{1,2}(\cdot)$  and  $\hat{F}_{3,2}(\cdot)$ . Similarly, let buyer 3 report  $\hat{v}_3, \hat{F}_{1,3}(\cdot)$ , and  $\hat{F}_{2,3}(\cdot)$ . Using  $\hat{F}_{1,2}(\cdot), \hat{F}_{2,3}(\cdot), \hat{F}_{3,2}(\cdot)$ , we can calculate (23) and values  $\hat{v}^*, \hat{p}_1, \hat{p}_2, \hat{p}_3$  as defined in the appendix. Then if buyer 1 reports  $\hat{v}_1 < \hat{v}^*$ , he pays

$$\int_0^{\hat{v}_1} x(\hat{F}_{2,3}(x)\hat{F}'_{3,2}(x) + \hat{F}'_{2,3}(x)\hat{F}_{3,2}(x)) dx$$

and wins if his reported valuation is highest; if buyer 1 reports  $v_1 \geq \hat{v}^*$ , he pays  $w$  and wins according to the lottery of property (ii) above, calculated using probabilities  $(\hat{p}_1, \hat{p}_2, \hat{p}_3)$ .

If buyers 2 and 3 report the probabilities truthfully, i.e.,

$$\hat{F}_{1,2}(\cdot) = F_1(\cdot), \quad \hat{F}_{2,3}(\cdot) = F_2(\cdot), \quad \hat{F}_{3,2}(\cdot) = F_3(\cdot),$$

then clearly it is optimal for buyer 1 to report  $\hat{v}_1$  truthfully. The pertinent question, therefore, is whether these probability distributions can be elicited truthfully (recall that the probability distributions are common knowledge among the buyers, and so buyers 2 and 3 are *capable* of revealing them; the only issue is whether or not they have the *incentive* to do so).

Fortunately, there is a simple classical method for inducing an agent to reveal probability distributions truthfully, namely, to confront him with a *scoring rule*. Suppose that an agent knows that outcomes  $x_1, \dots, x_n$  occur with probabilities  $q_1, \dots, q_m$ . Then, to get him to reveal this distribution, we can have him report  $\hat{q}_1, \dots, \hat{q}_m$  and pay him  $k \log \hat{q}_i$  if  $x_i$  occurs (where  $k$  is any positive constant). This means that the agent will choose  $(\hat{q}_1, \dots, \hat{q}_m)$  to maximize

$$k \sum q_i \log \hat{q}_i \quad \text{subject to} \quad \sum \hat{q}_i = 1.$$

It is immediate that the unique maximum occurs where  $\hat{q}_i = q_i$  for all  $i$ . In our auction setting, a scoring rule implies that if buyer  $i$  reports  $\hat{F}_j(\cdot)$  he should be paid  $k \log F'_j(\hat{v}_j)$  if buyer  $j$  reports  $\hat{v}_j$ .

These considerations suggest the following generalization of the all-pay auction:

- (i) buyer 1 reports  $\hat{v}_1, \hat{F}_{2,1}(\cdot), \hat{F}_{3,1}(\cdot)$ , buyer 2 reports  $\hat{v}_2, \hat{F}_{1,2}(\cdot), \hat{F}_{3,2}(\cdot)$ , buyer 3 reports  $\hat{v}_3, \hat{F}_{1,3}(\cdot), \hat{F}_{2,3}(\cdot)$ ;
- (ii)  $\hat{v}_1^*$  and  $(\hat{p}_{1,1}, \hat{p}_{2,1}, \hat{p}_{3,1})$  are computed using  $\hat{F}_{1,2}(\cdot), \hat{F}_{2,3}(\cdot)$ , and  $\hat{F}_{3,2}(\cdot)$ ,  $\hat{v}_2^*$  and  $(\hat{p}_{1,2}, \hat{p}_{2,2}, \hat{p}_{3,2})$  are computed using  $\hat{F}_{2,3}(\cdot), \hat{F}_{3,1}(\cdot)$ , and  $\hat{F}_{1,3}(\cdot)$ ,  $\hat{v}_3^*$  and  $(\hat{p}_{1,3}, \hat{p}_{2,3}, \hat{p}_{3,3})$  are computed using  $\hat{F}_{3,1}(\cdot), \hat{F}_{1,2}(\cdot)$ , and  $\hat{F}_{2,1}(\cdot)$ ;
- (iii) buyer  $i$  pays

$$\int_0^{\hat{v}_i} x(\hat{F}_{j,k}(x)\hat{F}_{k,j}(\hat{v}_j) + \hat{F}_{j,k}(\hat{v}_j)\hat{F}_{k,j}(x)) dx, \quad j \neq i \neq k;$$

- (iv) buyer  $i$  wins if  $\hat{v}_i > \max\{\hat{v}_j, \hat{v}_k\}$  and  $\hat{v}_i^* > \max\{\hat{v}_j, \hat{v}_k\}$ ,<sup>2</sup> he wins with probability  $\hat{p}_{i,i}/(\hat{p}_{i,i} + \hat{p}_{j,i})$  if  $\hat{v}_i, \hat{v}_j > \hat{v}_i^* > \hat{v}_k$ , he wins with probability  $\hat{p}_{i,i}$  if  $\hat{v}_i, \hat{v}_j, \hat{v}_k > \hat{v}_i^*$ ;
- (v) buyer  $i$  is paid  $K + k \log F'_{j,i}(\hat{v}_j) + k \log F'_{k,i}(\hat{v}_k)$ , where the payment can be made positive and as small as we like (assuming that  $F'_{j,i}(\hat{v}_j)$  and  $F'_{k,i}(\hat{v}_k)$  and bounded from above and below by positive numbers) by taking  $K$  and  $k$  sufficiently small.

---

<sup>2</sup> In equilibrium,  $\hat{v}_1^* = \hat{v}_2^* = \hat{v}_3^* = v^*$ . However, out of equilibria we could have, for example  $\hat{v}_i > \hat{v}_i^* > \hat{v}_j > \hat{v}_j^* > v_k$ , in which case, according to the allocation rules as applied to buyer  $i$ , buyer  $i$  should win (with probability 1), but as applied to buyer  $j$  (see the second line of (iv)) buyer  $j$  should win with positive probability  $\hat{p}_{j,j}/(\hat{p}_{i,j} + \hat{p}_{j,j})$ . Anomalies such as this can be dealt with, however, by defining  $\hat{v}^* = \min\{\hat{v}_1^*, \hat{v}_2^*, \hat{v}_3^*\}$  and using  $\hat{v}^*$  instead of  $\hat{v}_i^*$  and  $\hat{v}_j^*$ .

## 9. Possible extensions

The Groves (1973)/Clarke (1971) mechanism extends the second-price auction to more than one good. In future work, it would be interesting to investigate whether the same principles are applicable in our setting to permit an extension of the generalized all-pay auction to multiple goods.

We have focussed in this lecture on the case of *private values*. That is, we supposed that buyers *know* their valuations. However, Dasgupta and Maskin (1999) show that the second-price auction can be extended to deal with the possibility that one buyer's valuation might depend on others' private information. It is also worth seeing whether the generalized all-pay auction can be extended to accommodate this possibility too.

## Acknowledgements

I should like to thank the NSF for research support.

## Appendix

In the case of three buyers,  $v^*$  and  $(p_1, p_2, p_3)$  are determined by the equations

$$\begin{aligned} & \left[ p_1(1 - F_2(v^*))(1 - F_3(v^*)) + \frac{p_1}{p_1 + p_2}(1 - F_2(v^*))F_3(v^*) \right. \\ & \quad \left. + \frac{p_1}{p_1 + p_3}(1 - F_3(v^*))F_2(v^*) + F_2(v^*)F_3(v^*) \right] v^* - w \\ & = F_2(v^*)F_3(v^*)v^* - \int_0^{v^*} x(F_2(x)F_3'(x) + F_2'(x)F_3(x)) dx, \\ & \left[ (p_2(1 - F_1(v^*))(1 - F_3(v^*)) + \frac{p_2}{p_2 + p_1}(1 - F_1(v^*))F_3(v^*) \right. \\ & \quad \left. + \frac{p_2}{p_2 + p_3}(1 - F_3(v^*))F_1(v^*) + F_1(v^*)F_3(v^*) \right] v^* - w \\ & = F_1(v^*)F_3(v^*)v^* - \int_0^{v^*} x(F_1(x)F_3'(x) + F_1'(x)F_3(x)) dx, \\ & \left[ p_3(1 - F_1(v^*))(1 - F_2(v^*)) + \frac{p_3}{p_1 + p_3}(1 - F_1(v^*))F_2(v^*) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{p_3}{p_2 + p_3} (1 - F_2(v^*)) F_1(v^*) + F_1(v^*) F_2(v^*) \Big] v^* - w \\
& = F_1(v^*) F_2(v^*) v^* - \int_0^{v^*} x (F_1(x) F_2'(x) + F_1'(x) F_2(x)) dx.
\end{aligned}$$

## References

- Che, Y.K., Gale, J., 1996. Expected revenue of the all-pay auctions and first-price sealed-bid auctions with budget constraints. *Economics Letters* 50, 373–380.
- Clarke, E., 1971. Multipart pricing of public goods. *Public Choice* 11, 17–33.
- Dasgupta, P., Maskin, E., 1999. Efficient auctions. *Quarterly Journal of Economics*, forthcoming.
- Groves, T., 1973. Incentives in teams. *Econometrica* 41, 41–63.
- Laffont, J.-J., Robert, J., 1996. Optimal auction with financially constrained buyers. *Economics Letters* 52, 181–186.
- Vickrey, W., 1961. Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance* 16, 8–37.