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The Existence of Equilibrium in Discontinuous Economic Games, I: Theory

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1. INTRODUCTION

In this paper and its sequel (Dasgupta and Maskin (1986)) we study the existence of Nash equilibrium in games where agents' payoff (or utility) functions are discontinuous. Our enquiry is motivated by a number of recent studies that have uncovered serious existence problems in seemingly innocuous economic games. These examples include models of spatial competition (Eaton and Lipsey (1975) and Shaked (1975)), Hotelling's model of price competition (d'Aspremont, Gabszewicz and Thisse (1979)), and models of market-dependent information (e.g. the insurance-market example of Rothschild and Stiglitz (1976) and Wilson (1977)). In fact, the non-existence of Nash equilibrium in simple economic models was noted long ago by Edgeworth (1925) in his critique of Bertrand's (1883) analysis of price setting duopolists; (see Chamberlin (1956), pp. 34-46).

These examples can be readily cast as games in normal form with continuous strategy sets (see the sequel to this paper). Since they fail in general to possess an equilibrium they must obviously violate one or more of the hypotheses of the classical existence theorem for games of this type (e.g. Debreu (1952), Glicksberg (1952), and Fan (1952)). These hypotheses typically include *continuity* and a limited form of *quasi-concavity* of the payoff functions (in addition to the usual convexity and compactness assumptions on strategy sets). To better understand a model that does not possess an equilibrium it is helpful to identify the failure—violation of quasi-concavity or of continuity—to which the non-existence can be attributed. Drawing such a distinction allows us to construct a taxonomy of "equilibrium failure" and to highlight the structural similarities or differences among models. It also enables us to see how the models may be plausibly modified to restore the existence of equilibrium. This is important because the failure of an economic model to possess an equilibrium has sometimes led economists to cast doubt on the validity—and even the internal consistency—of the construct.

In the sequel to this paper we shall see that the utility functions in the economic games referred to earlier are neither continuous nor quasi-concave.[1] However, we demonstrate that the payoff functions in mildly modified versions of these constructs exhibit two weaker forms of continuity which, together with the requirement of quasi-concavity, suffice for the existence of an equilibrium (Theorem 2, below). From this we may conclude that, at least in the modified versions of these models, discontinuities in the payoff functions are not the real source of the problem. Rather, it is the failure of the payoff functions to be quasi-concave which is "responsible" for the non-existence of equilibrium.

These observations bear on the existence of Nash equilibrium in *pure strategies.* In this study, however, we are concerned, in the main, with the existence of Nash equilibrium in *mixed (random) strategies.* For games in which payoff functions are not quasi-concave, mixed strategies may offer an escape route from the problem of non-existence. Yet, the classical existence theorems on mixed-strategy equilibrium (e.g. Glicksberg (1952)) typically hypothesize continuous utility functions, and are, therefore, not applicable to the economic games we have mentioned. In this study we show that the classical theorems can be suitably generalized. In the sequel to this paper we observe that a significant feature shared by the simplest versions of most of these economic models is that their discontinuities occur only at points where two or more agents use the *same* strategy. We also note that even in the more general versions of these models the set of discontinuities is of dimension lower than that of the strategy space. Thus, in particular, the set of discontinuities is of (Lebesgue) measure zero. In this paper we establish equilibrium existence theorems (Theorems 4 and 5) for games possessing discontinuity sets of this kind. Theorem 4 examines a sequence of successively finer finite approximations of a game with a continuous strategy set. The theorem asserts that if the limiting strategies of a sequence of equilibrium strategy vectors for these finite games are themselves atomless on the closure of the set of discontinuities, then they constitute an equilibrium (mixed) strategy vector for the game in question.

Another important feature shared by these economic models is that, although agents' utility functions are discontinuous, their sum is an upper semi-continuous function of strategies. (This is not quite true of all the examples to be studied in the sequel, but nearly enough to suffice for our purpose.) Moreover, at the points of discontinuity, the utility functions satisfy some limited form of *lower* semi-continuity. Theorem 5, our main result, states, roughly speaking, that with a discontinuity set of the type mentioned above, an (upper semi) continuous-sum game in which individual utility functions satisfy a weak form of lower semi-continuity possesses a mixed strategy equilibrium. Theorem 5a extends our main result to cases in which a player has a strategy where his payoff function is discontinuous regardless of other agents' strategies. Theorem 5b shows that we may be able to dispense with the requirement that the sum of payoffs be upper semi-continuous, if a downward jump in one player's payoff is always accompanied by an upward jump in another player's payoff. (See also the interesting paper by Simon (1984), who calls such pairs of jumps "complementary discontinuities".) In addition to the assumptions of Theorem 5, Theorem 6 hypothesizes that the game is symmetric. The theorem asserts the existence of a symmetric equilibrium—one in which all agents choose the same probability measure. More important, it also provides conditions under which an equilibrium measure is atomless on the discontinuity set.

In the sequel to this paper, where we explore economic applications of these results, we shall appeal to Theorems 5 and 6 to establish the existence and some of the qualitative features of mixed-strategy equilibrium in the economic models mentioned above.

The idea of exploring mixed-strategy equilibria in discontinuous games has occurred to many. The only general existence theorem that we are aware of is due to Glicksberg (1950) who showed that a two-person zero-sum game on the unit square always has a minimax value if the payoff function is either upper or lower semi-continuous (see Definitions 2 and 4 below). In all the other studies known to us the approach has been to demonstrate the existence of equilibrium by construction. Thus Karlin (1959) has analysed what are called "games of timing", which are discontinuous two-person zero-sum games.[2] In the economics literature Beckmann (1965) has demonstrated, by construction, the existence of a mixed-strategy equilibrium in the symmetric version of the Bertrand-Edgeworth model with a linear market demand curve. (See also Shubik (1955) and Shapley (1957)). More recently Shilony (1977) and Varian (1980), also constructively, have demonstrated the existence of mixed-strategy equilibrium in certain symmetric models of price-setting firms. The limitations of these constructive approaches are fairly obvious. Not only are the existence results necessarily specific rather than general, but one fails to learn why the discontinuities in the model (which are after all, what is crucial) do not prevent the existence of equilibrium.[3] Furthermore, the demonstrations, so far as we are aware, apply in the main to symmetric games; and it is not clear that they extend to asymmetric versions. It bears emphasis that Theorems 4 and 5, our main existence theorems, do not assume that the game is symmetric.

The plan of this paper is as follows. In Section 2 we introduce much of the notation and present two theorems (Theorems 1 and 2) on the existence of pure-strategy equilibrium. Theorem 1, due to Debreu (1952), is well known. Theorem 2, which postulates only a weak form of continuity for the utility functions, is a straightforward extension. We have included it because the payoff functions of many economic models, including slightly modified versions of the ones mentioned above, fail to satisfy continuity but are continuous in our weaker sense (see the sequel). Theorem 2 thus places the "blame" for the non-existence of pure-strategy equilibrium squarely on the lack of quasi-concavity of these payoff functions. The fact that non-existence can be ascribed to a failure of quasi-concavity, and not to continuity, makes the existence of mixed-strategy equilibrium more plausible. In Section 3 mixed-strategies are introduced formally, and we state Glicksberg's (1952) theorem for continuous payoff functions as Theorem 3. We then provide an example (due to Sion and Wolfe (1957)) of a game with discontinuous payoff functions that does not possess even an ε-equilibrium. In Section 4 we present existence results (Theorems 4, 5, 5a and 5b) for games with discontinuous payoff functions. Before proving them in detail in Sub-sections 4.3 and 4.4, we provide an outline of the argument in Sub-section 4.2. These theorems have been strongly motivated by the economic models of the sequel. Nevertheless, we believe that they cannot be generalized much further. Indeed, as the examples of Section 6 demonstrate, the theorems are "tight", in the sense that dropping any of their hypotheses renders them invalid. Finally, Section 5 treats symmetric games.

To simplify the exposition and notation, we suppose in Sections 4–6 that the strategy set of each agent is one-dimensional. In fact, this is so in all the economic examples mentioned earlier, except for the two-dimensional location model of Shaked (1975) and the insurance-market example of Rothschild, Stiglitz and Wilson. In the Appendix we present the multi-dimensional generalizations of Theorems 4, 5, and 6.

2. EQUILIBRIUM IN PURE STRATEGIES

Agents are assumed to be N in number, indexed by i or $j = 1, \dots, N$. $A_i \subseteq R^m$ denotes the set of feasible actions (strategies) from which agent i may choose.[4] A typical element of A_i is a_i. Write $\boldsymbol{a}$ for an N-tuple of actions. Thus $\boldsymbol{a} \in \prod_{j=1}^N A_j$. Define $\boldsymbol{a}_{-i} \equiv (a_1, a_2, \dots,, a_{i-1}, a_{i+1}, \dots, a_N)$ as the vector of actions of all agents other than i. We shall often write $\boldsymbol{a} = (a_i, \boldsymbol{a}_{-i})$, $A = \prod_{j=1}^N A_j$ and $A_{-i} = \prod_{j \neq i} A_j$.

Let $U_i : \prod_{j=1}^N A_j \to R^1$ be the *payoff* (*utility*) *function* of agent i. A *game* is summarized as $[(A_i, U_i);\ i = 1, \dots, N]$, where A_i is agent i's feasible set of (pure) strategies, and U_i is his payoff (utility) function. Define $\psi_i(\boldsymbol{a}_{-i}) = \{a_i \in A_i \mid U_i(a_i, \boldsymbol{a}_{-i}) = \max_{a_i' \in A_i} U_i(a_i', \boldsymbol{a}_{-i})\}$. $\psi_i(\boldsymbol{a}_{-i})$ is called agent i's reaction correspondence.

Definition 1. $\boldsymbol{a}^* \in A$ is a Nash equilibrium in pure strategies of the game $[(A_i, U_i); i = 1, \ldots, N]$ if $U_i(a_i^*, \boldsymbol{a}_{-i}^*) = \max_{a_i \in A_i} U_i(a_i, \boldsymbol{a}_{-i}^*)$ for all $i = 1, \ldots, N$; (or, equivalently if $\boldsymbol{a}^* \in \prod_{i=1}^N \psi_i(\boldsymbol{a}_{-i}^*)$).

The following is a basic result.

Theorem 1 (Debreu, Glicksberg, Fan). *Let $A_i \subseteq R^m$, $(i = 1, \ldots, N)$ be non-empty, convex and compact. If, $\forall i$, $U_i: A \to R^1$ is continuous in $\boldsymbol{a}$ and quasi-concave in a_i, the game $[(A_i, U_i); i = 1, \ldots, N]$ possesses a pure-strategy Nash equilibrium.*

In the sequel we shall note that each of the economic models mentioned in the Introduction violates both the continuity and quasi-concavity requirements. This prompts us to extend Theorem 1 so that it applies to games with a limited form of continuity.

In what follows we shall always suppose that A_i $(i = 1, \ldots, N)$ is non-empty and compact.

Definition 2. $U_i: A \to R^1$ is *upper semi-continuous* (u.s.c.) if for any sequence

$$\{\boldsymbol{a}^n\} \subseteq A \text{ such that } \boldsymbol{a}^n \to \boldsymbol{a}, \limsup_{n\to\infty} U_i(\boldsymbol{a}^n) \leqq U_i(\boldsymbol{a}).$$

Definition 3. $U_i: A \to R^1$ is *graph-continuous* if for all $\bar{\boldsymbol{a}} \in A$ there exists a function $F_i: A_{-i} \to A_i$ with $F_i(\bar{\boldsymbol{a}}_{-i}) = \bar{a}_i$ such that $U_i(F_i(\boldsymbol{a}_{-i}), \boldsymbol{a}_{-i})$ is continuous at $\boldsymbol{a}_{-i} = \bar{\boldsymbol{a}}_{-i}$.

We call the property described in Definition 3 graph continuity, because, if one graphs a player's payoff as a function of his own strategy (holding the strategies of the other players fixed), and if this graph changes continuously as one varies the strategies of the other players, then the player's payoff function is graph continuous in the sense of Definition 3. (Actually, graph continuity, as we have defined it, is a bit weaker than this.)

We can now state and prove

Theorem 2. *Let $A_i \subseteq R^m$ $(i = 1, \ldots, N)$, be non-empty, convex and compact. If $\forall i$, $U_i: A \to R^1$ is quasi-concave in a_i, upper semi-continuous in $\boldsymbol{a}$ and graph-continuous, then the game $[(A_i, U_i); i = 1, \ldots, N]$ possesses a pure-strategy Nash equilibrium.*

Proof. We prove the Theorem for the case $N = 2$: (extension to $N > 2$ is immediate). Let ψ_1 and ψ_2 be the reaction correspondences of the two agents. They exist and are compact-valued because U_1 and U_2 are upper semi-continuous. Moreover, they are convex valued because U_1 and U_2 are quasi-concave in a_1 and a_2 respectively. We need now only verify that the ψ_i's are upper hemi-continuous correspondences.[5]

Consider the sequences $\{a_2^n\} \subseteq A_2$ and $\{a_1^n\} \subseteq A_1$ such that $a_2^n \to \bar{a}_2$, $a_1^n \to \bar{a}_1$ and $\forall n$, $a_1^n \in \psi_1(a_2^n)$. If $\bar{a}_1 \notin \psi_1(\bar{a}_2)$ there exists $a_1^* \in A_1$ such that $U_1(a_1^*, \bar{a}_2) > U_1(\bar{a}_1, \bar{a}_2)$. Let $\varepsilon = [U_1(a_1^*, \bar{a}_2) - U_1(\bar{a}_1, \bar{a}_2)]/2$. Then there exists F with $F(\bar{a}_2) = a_1^*$ and $\delta > 0$ such that $\|a_2 - \bar{a}_2\| < \delta$ implies $|U_1(F_1(a_2), a_2) - U_1(a_1^*, \bar{a}_2)| < \varepsilon$. It follows that, for n sufficiently large, $|U_1(F_1(a_2^n), a_2^n) - U_1(a_1^*, \bar{a}_2)| < \varepsilon$. But $U_1(F_1(a_2^n), a_2^n) \leqq U_1(a_1^n, a_2^n)$. Therefore, $U_1(a_1^n, a_2^n) \geqq U_1(F_1(a_2^n), a_2^n) \geqq U_1(\bar{a}_1, \bar{a}_2) + \varepsilon$ for n sufficiently large. But $U_1(\bar{a}_1, \bar{a}_2) \geqq \limsup_{n\to\infty} U_1(a_1^n, a_2^n)$, a contradiction. Therefore, $\bar{a}_1 \in \psi_1(\bar{a}_2)$, and so ψ_1 is upper hemi-continuous. Similarly for ψ_2. We may therefore apply the Kakutani fixed point theorem to the correspondence $\psi_1 \times \psi_2$ to conclude that a Nash equilibrium exists. ||

In fact we can obtain a result very similar to Theorem 2 by recalling the following:

Definition 4. A function $g: A \to R^1$ is *lower semi-continuous* (l.s.c.) if for all sequences $\{a^n\} \subseteq A$ such that $\boldsymbol{a}^n \to \bar{\boldsymbol{a}}$, $\liminf_{n\to\infty} g(\boldsymbol{a}^n) \geqq g(\bar{\boldsymbol{a}})$.

We now have the following

Corollary. *$\forall i$, let $A_i \subseteq R^m$, $(i = 1, \ldots, N)$ be non-empty, convex and compact, and let $U_i: A \to R^1$ be quasi-concave in a_i and u.s.c. Define $V_i(\boldsymbol{a}_{-i}) = \max_{a_i} U_i(a_i, \boldsymbol{a}_{-i})$; ($V_i$ is well-defined because U_i is u.s.c.). If $\forall i$, V_i is l.s.c., then the game $[(A_i, U_i);\ i = 1, \ldots, N]$ possesses a pure-strategy Nash equilibrium.*[6]

Proof. Virtually identical to the proof in Theorem 2. ||

In the sequel we will note that the payoff functions in the models of product and price competition referred to earlier here violate both upper semi-continuity and quasi-concavity, although they satisfy graph-continuity. We will also observe that it is possible to modify the payoff functions slightly, and plausibly, so that the resulting payoff functions satisfy the u.s.c. requirement. Nevertheless, even in such modified versions, an equilibrium in pure strategies fails to exist in general. Theorem 2 suggests, therefore, that the explanation for the non-existence in these modified examples is solely the lack of quasi-concavity; the failure of the payoff functions to be continuous is irrelevant. We turn therefore to mixed strategies to convexify the payoff functions. In Section 4 we develop two theorems, one of which (Theorem 5) is used in the sequel to demonstrate the existence of mixed-strategy equilibrium in each of the models cited in the Introduction.

3. NASH EQUILIBRIUM IN MIXED STRATEGIES

Let $A_i \subseteq R^m$, $(i = 1, \ldots, N)$, be non-empty, convex and compact. Let $U_i = A \to R^1$ be bounded and measurable for all i. Let $D(A_i)$ be the space of all (Borel) probability measures on A_i.[7]

Definition 5. A mixed-strategy (Nash) equilibrium of the game $[(A_i, U_i); i = 1, \ldots, N]$ is an N-tuple of probability measures $(\mu_1^*, \ldots, \mu_i^*, \ldots, \mu_N^*)$, with $\mu_i^* \in D(A_i)$, such that for all $i = 1, \ldots, N$

$$\int U_i(a_i, \boldsymbol{a}_{-i}) d(\mu_i^*(a_i) \times \boldsymbol{\mu}_{-i}^*(\boldsymbol{a}_{-i})) = \max_{\mu_i \in D(A_i)} \int U_i(a_i, \boldsymbol{a}_{-i}) d(\mu_i(a_i) \times \boldsymbol{\mu}_{-i}^*(\boldsymbol{a}_{-i})),$$

where $\boldsymbol{\mu}_{-i}^*(\boldsymbol{a}_{-i}) \equiv \prod_{j \neq i} \mu_j^*(a_j)$.

In what follows we shall often denote by $\boldsymbol{\mu}$ the product measure $\mu_1 \times \cdots \times \mu_N$. That is, if $E \subseteq A$ is a (Borel) measurable set, then we write $\boldsymbol{\mu}(E) = (\mu_1 \times \cdots \times \mu_N)(E)$. Likewise, we shall often write

$$\int_A U_i(\boldsymbol{a}) d\boldsymbol{\mu} = \int_A U_i(a_1, \ldots, a_N) d(\mu_1(a_1) \times \cdots \times \mu_N(a_N)),$$

and also

$$\int_{A_{-i}} U_i(\tilde{a}_i, \boldsymbol{a}_{-i}) d\boldsymbol{\mu}_{-i} = \int_{A_{-i}} U_i(\tilde{a}_i, \boldsymbol{a}_{-i}) d(\mu_1(a_1) \times \cdots \times \mu_{i-1}(a_{i-1}) \times \mu_{i+1}(a_{i+1}) \times \cdots \times \mu_N(a_N)).$$

Where there is no risk of confusion we shall not explicitly indicate the domain over which the integration is being carried out. Thus, for $\int U_i(\boldsymbol{a})d\boldsymbol{\mu}$ it will be understood that the domain is $A \equiv \prod_{i=1}^{N} A_i$.

Now, we know from Nash (1950) that if A_i is a *finite* set a mixed-strategy equilibrium exists. For our purposes the more relevant existence theorem is

Theorem 3 (Glicksberg (1952)). *Let $A_i \subseteq R^m$, $(i=1, \ldots, N)$ be non-empty and compact. Let $U_i: A \rightarrow R^1$ be continuous. Then there exists a mixed-strategy equilibrium for the game* $[(A_i, U_i); i=1, \ldots, N]$.

The remainder of this paper is devoted to extending Theorem 3 by relaxing the requirement that payoff functions be continuous. The theorems that follow are motivated by the economic models to be analysed in the sequel. In these models we shall note that payoff functions are bounded, and are in fact continuous except on a set of measure zero (with respect to the Lebesgue measure) in the strategy space. One might conjecture that such games are assured of at least an ε-equilibrium in mixed strategies, for all positive ε.[8] That this speculation is ill-founded is borne out by

Example 1 (Sion and Wolfe (1957)). Consider a two-person zero-sum game played on the unit square. Thus $A_1 = A_2 = [0, 1]$. The payoff $U_1(a_1, a_2)$ to agent 1 is given by the function

$$U_1(a_1, a_2) = \begin{cases} -1 & \text{if } a_1 < a_2 < a_1 + \frac{1}{2} \\ 0 & \text{if } a_1 = a_2 \text{ or } a_2 = a_1 + \frac{1}{2} \\ +1 & \text{otherwise.} \end{cases} \tag{1}$$

(See Figure 1 below.)

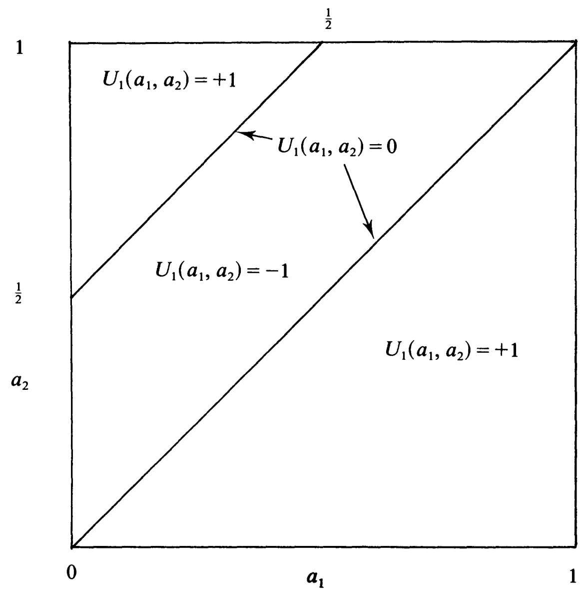

FIGURE 1

Define

$$V_1 \equiv \sup_{\mu_1 \in D([0,1])} \inf_{\mu_2 \in D([0,1])} \int U_1(a_1, a_2) d\mu_1 d\mu_2$$

and

$$V_2 \equiv \inf_{\mu_2 \in D([0,1])} \sup_{\mu_1 \in D([0,1])} \int U_1(a_1, a_2) d\mu_1 d\mu_2$$

as the "values" of the game for the two players. It can in fact be shown that $V_1 = \frac{1}{3}$ and $V_2 = \frac{3}{7}$, establishing therefore that no equilibrium exists and, indeed, that for ε sufficiently small not even an ε-equilibrium exists. It can also be confirmed that the value pair $(\frac{1}{3}, \frac{3}{7})$ is attained if the mixed strategy chosen by the first agent puts probability weight only on the points $\{0, \frac{1}{2}, 1\}$, and that of the second agent puts weight only on $\{\frac{1}{4}, \frac{1}{2}, 1\}$.

4. THE EXISTENCE OF MIXED-STRATEGY EQUILIBRIUM

4.1. *Characterization of discontinuity set*

Theorem 3 establishes that the non-existence of equilibrium in Example 1 is due to discontinuities in the payoff functions. Note that the discontinuities occur only when either the players' strategies are the same, or they differ by $\frac{1}{2}$: that is, they occur on a set of dimension one less than that of the joint strategy space. But this is precisely the kind of discontinuity set possessed by the economic models referred to earlier, which, we shall show in the sequel to this paper, *do* have equilibria. We must, therefore, investigate what distinguishes these economic games from Example 1. Theorems 4 and 5 below make the differences clear.

Let $[(A_i, U_i); i = 1, \ldots, N]$ be a game, where $A_i \subseteq R^1$ is a closed interval.[9] In what follows, we shall suppose that for each i, the discontinuities of U_i are confined to a subset of a continuous manifold of dimension less than N. To be precise, for each pair of agents $i, j \in \{1, \ldots, N\}$, let $D(i)$ be a positive integer, and for each integer d, with $1 \leqq d \leqq D(i)$, let $f_{ij}^d \colon R^1 \to R^1$ be a *one-to-one, continuous* function. Finally, for each $i \in \{1, \ldots, N\}$ define

$$A^*(i) = \{(a_1, \ldots, a_N) \in A \mid \exists j \neq i, \exists d, 1 \leqq d \leqq D(i) \text{ such that } a_j = f_{ij}^d(a_i)\}. \tag{2}$$

We shall suppose that the discontinuities of $U_i(\boldsymbol{a})$ are confined to a subset $A^{**}(i)$ of $A^*(i)$. In other words, $A^{**}(i)$ is the set of discontinuities of $U_i(\boldsymbol{a})$, *and* $\forall i$, $A^{**}(i) \subseteq A^*(i)$. Note in particular from (2) that $A^*(i) \subseteq R^N$ is of (Lebesgue) measure zero. Finally, $\forall a_i \in A_i$ define $A_{-i}^*(a_i) = \{\boldsymbol{a}_{-i} \in A_{-i} \mid (a_i, \boldsymbol{a}_{-i}) \in A^*(i)\}$, and $A_{-i}^{**}(a_i) = \{\boldsymbol{a}_{-i} \in A_{-i} \mid (a_i, \boldsymbol{a}_{-i}) \in A^{**}(i)\}$.

Observe that the discontinuity set is defined so that a player's payoff is discontinuous only when his strategy is "related" to that of another (through the function f_{ij}^d). In many of the economic examples mentioned, f_{ij}^d is the identity function, so that discontinuities occur only when two players' strategies coincide. Notice that for the game in Example 1, we may take $D(1) = D(2) = 2$, $f_{12}^1(a_1) = a_1$, $f_{12}^2(a_1) = a_1 + \frac{1}{2}$, and define $A^*(i)$ by (2). In this case $A^*(1)$ is the same as $A^*(2)$, i.e. the union of the two 45° lines in the diagram accompanying Example 1. In fact, each point of $A^*(i)$ is a point of discontinuity of U_i; i.e. $A^{**}(i) = A^*(i)$.

Note that the requirement that the function f_{ij}^d be one-to-one does not rule out a non-monotonic curve of discontinuities; since each monotonic subcurve can be made to correspond to a different f_{ij}^d. The real force of the assumption is to exclude f_{ij}^d's that are "vertical" or "horizontal" on some interval. (See Example 4 of Section 6 for the explanation of why this restriction is in some sense necessary. See also Theorem 5a below.)

4.2. *An outline of the argument*

Consider a game $[(A_i, U_i); i=1,\dots,N]$, where $A_i \subseteq R^1$ is a closed interval, and suppose that $U_i(\boldsymbol{a})$ is bounded, and is continuous except on a subset, $A^{**}(i)$, of $A^*(i)$, as defined by (2). We wish to state conditions under which the game possesses a mixed-strategy equilibrium. In this sub-section we provide an idea of the arguments we use to prove our main existence theorem (Theorem 5).

Consider a sequence of finite approximations (lattices) of A_i. We denote the n-th finite approximation by $A_i^n (n>0)$, and we suppose that the lattice is finer the larger the value of n, and that $A_i^n \to A_i$ as $n \to \infty$. We denote the finite game defined on the n-th lattice by $[(A_i^n, U_i); i=1,\dots,N]$, where it is understood that for the finite game the payoff function of agent i is the restriction of U_i to the set $\prod_{i=1}^N A_i^n$. By Nash's theorem we know that this game possesses a mixed-strategy equilibrium. Let $(\mu_1^n,\dots,\mu_i^n,\dots,\mu_N^n)=\boldsymbol{\mu}^n$ be an equilibrium. Since the space of (Borel) probability measures is sequentially compact under the topology of weak convergence, we may as well suppose that the sequence $\{\boldsymbol{\mu}^n\}$ itself converges. Let $(\mu_1^*,\dots,\mu_i^*,\dots,\mu_N^*)=\boldsymbol{\mu}^*$ be the limit of this sequence; i.e. $\lim_{n\to\infty}\boldsymbol{\mu}^n=\boldsymbol{\mu}^*$.[10] We would like to ensure that $\boldsymbol{\mu}^*$ is in fact a mixed-strategy equilibrium of the limiting game $[(A_i, U_i); i=1,\dots,N]$.

Let $\bar{A}^{**}(i)$ denote the closure of $A^{**}(i)$. Since $A^*(i)$ is closed we conclude that $\bar{A}^{**}(i)\subseteq A^*(i)$. Let $A_i^{**}(i)$ (resp. $\bar{A}_i^{**}(i)$, $A_i^*(i)$) be the projection of $A^{**}(i)$ (resp. $\bar{A}^{**}(i)$, $A^*(i)$) onto A_i. (Thus, for example, $\forall \tilde{a}_i \in A_i^{**}(i)$, $\exists \boldsymbol{a}_{-i}\in A_{-i}$ such that $(\tilde{a}_i, \boldsymbol{a}_{-i})$ is a point of discontinuity of U_i.) Theorem 4 says that if $\forall i$, μ_i^* is atomless on $\bar{A}_i^{**}(i)$, then $\boldsymbol{\mu}^*$ is an equilibrium of $[(A_i, U_i); i=1,\dots,N]$. This is intuitively appealing. If for large enough n, the equilibrium mixed-strategy, μ_i^n, places very little probability weight on points that lie in $\bar{A}_i^{**}(i)$, then the fact that U_i is discontinuous at points in $A^{**}(i)$ does not matter for the limiting game.

But Theorem 4 is not very helpful in determining whether a game has an equilibrium: in order to verify whether μ_i^* is atomless on $\bar{A}_i^{**}(i)$ we need to compute μ_i^n. It is desirable to put conditions directly on the payoff functions, U_i, that guarantee that the limiting vector of measures, $\boldsymbol{\mu}^*$, is an equilibrium. This is what Theorem 5 does.

In addition to the requirement that the discontinuity sets, $A^{**}(i)$, be contained in $A^*(i)$, where $A^*(i)$ is defined by (2), Theorem 5 requires that the *sum* of the utility functions be *upper* semi-continuous (see Definition 2 above), and that each player's utility function satisfies a weak form of *lower* semi-continuity as a function of his own strategy (see Definition 6 below). We now present a road-map of the arguments leading to Theorem 5.

Once again, consider a sequence, $\{\boldsymbol{\mu}^n\}$, of equilibria in successively finer finite approximations to the game $[(A_i, U_i); i=1,\dots,N]$. Let $\boldsymbol{\mu}^*=\lim_{n\to\infty}\boldsymbol{\mu}^n$. Because U_i is bounded and A_i is compact, the sequence $\{\int U_i(\boldsymbol{a})d\boldsymbol{\mu}^n\}$ has a convergent subsequence. Without loss of generality, therefore, we may assume that it converges itself. Via Lemmas 1-3 we will show that for any given player i there *cannot* be more than a *countable* number of pure strategies, $\bar{a}_i$, such that

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})d\boldsymbol{\mu}_{-i}^* > \lim_{n\to\infty}\int U_i(\boldsymbol{a})d\boldsymbol{\mu}^n. \tag{3}$$

(This demonstration relies only on the fact that U_i is continuous *off* the set $A^{**}(i)$, where $A^{**}(i)$ is a subset of $A^*(i)$, defined by (2). It does not require that the game be an upper-semi-continuous-sum game, nor that U_i satisfy the weak form of lower-semi-continuity.) Now, of course, Lemmas 1–3 are only an intermediate stage of the argument, because we will wish finally to prove that there exists *no* $\bar{a}_i$ such that (3) holds. In order to prove this we will need the further conditions on the payoff functions that we have alluded to. To see this, note that because $\sum_{i=1}^N U_i(\boldsymbol{a})$ is upper-semi-continuous, we have

$$\lim_{n\to\infty} \int \sum U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n \leqq \int \sum U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^*, \tag{4}$$

and therefore, $\exists i$ such that

$$\lim_{n\to\infty} \int U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n \leqq \int U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^*. \tag{5}$$

From (5) one may conclude that $\exists \tilde{a}_i \in A_i$ such that

$$\lim_{n\to\infty} \int U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n \leqq \int U_i(\tilde{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}^*_{-i}. \tag{6}$$

In Lemma 4 it is argued that (5) must hold with *equality* for this *i*. For if not, then $\exists \tilde{a}_i \in A_i$ such that (6) is a strict inequality. Therefore, the fact that U_i is weakly lower semi-continuous in a_i (see Definition 6 below), would imply that there is an interval in A_i containing $\tilde{a}_i$ such that (3) holds for all $\bar{a}_i$ in this interval. But this contradicts the conclusion that (3) can hold for only countably many $\bar{a}_i$. Thus (5) must be an equality for all *i*.[11] Theorem 5 establishes that $\boldsymbol{\mu}^*$ is an equilibrium of the limiting game. For if it is not, then $\exists i$ and $\exists \tilde{a}_i \in A_i$ such that

$$\int U_i(\tilde{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}^*_{-i} > \int U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^*.$$

But this, in conjunction with the fact that (5) holds with equality implies that (6) holds with strict inequality, generating the same contradiction as before.

4.3. *Equilibrium when points of discontinuity are not atoms*

In this sub-section we prove Theorem 4 via two lemmas that are also useful in the proof of Theorem 5. Lemma 1 is straightforward, but its proof is tedious.

Lemma 1. *Let $A_i \subseteq R^1$, $(i = 1, \ldots, N)$, be a closed interval, and let μ_i be a (Borel) probability measure defined on A_i. Let $A^*(i) \subseteq A$ be defined by* (2), *and let $A^{**}(i)$ be a subset of $A^*(i)$. If $\forall i$, μ_i is atomless on $A_i^{**}(i)$ (i.e. $\forall a_i \in A_i^{**}(i)$, $\mu_i(\{a_i\}) = 0$), then $\forall i$, $\boldsymbol{\mu}(A^{**}(i)) = 0$. Furthermore, if for $\bar{a}_i \in A_i$, $\mu_j(\{f_{ij}^d(\bar{a}_i)\}) = 0$, $\forall j \neq i$ and $\forall d$, $1 \leqq d \leqq D(i)$, then $\boldsymbol{\mu}_{-i}(A^*_{-i}(\bar{a}_i)) = 0$, where $A^*_{-i}(\bar{a}_i) \equiv \{\boldsymbol{a}_{-i} \in A_{-i} \mid (\bar{a}_i, \boldsymbol{a}_{-i}) \in A^*(i)\}$.*

Proof. Note that $\forall i$, we have

$$\begin{aligned}\boldsymbol{\mu}(A^{**}(i)) &\equiv (\mu_1 \times \cdots \times \mu_N)(A^{**}(i)) \\ &\leqq \textstyle\sum_{s\neq i} \sum_{1\leqq d\leqq D(i)} (\mu_1 \times \cdots \times \mu_N)(\{(a_1, \ldots, a_N) \in A \mid a_s = f_{is}^d(a_i)\} \cap A^{**}(i)).\end{aligned} \tag{7}$$

We now construct a lattice on $A^*(i)$ as follows: Let $a_i^* \equiv \min A_i$ and $a_i^{**} \equiv \max A_i$. For each i, j and d as above, each positive integer n, and $r \in \{0, \ldots, (n-1)\}$, let

$$B_i^n(r) \equiv A_i^{**}(i) \cap \left\{ a_i \in A_i \,\middle|\, a_i^* + \frac{r}{n}(a_i^{**} - a_i^*) \leqq a_i \leqq a_i^* + \frac{(r+1)}{n}(a_i^{**} - a_i^*) \right\},$$

and

$$B_j^n(r, i, d) \equiv A_j^{**}(j) \cap \left\{ a_j \in A_j \,\middle|\, f_{ij}^d\left(a_i^* + \frac{r}{n}(a_i^{**} - a_i^*) \right) \leqq a_j \leqq f_{ij}^d\left(a_i^* + \left(\frac{r+1}{n}\right)(a_i^{**} - a_i^*) \right) \right\}.$$

Because $\forall i$, μ_i is atomless on $A_i^{**}(i)$ and because of the continuity of the functions f_{ij}^d, $\forall \varepsilon > 0$, $\exists \bar{n}$ such that $\forall n > \bar{n}$, $\forall j$, $\forall d$ with $1 \leqq d \leqq D(i)$, and $\forall r \in \{0, \ldots, n-1\}$,

$$\mu_i(B_r^n(r)) < \varepsilon \quad \text{and} \quad \mu_j(B_j^n(r, i, d)) < \varepsilon. \tag{8}$$

From (7) it follows that

$$\boldsymbol{\mu}(A^{**}(i)) \leqq \textstyle\sum_{j \neq i} \left\{ \sum_{1 \leqq d \leqq D(i)} \sum_{r=0}^{n-1} \mu_i(B_i^n(r)) \mu_j(B_j^n(r, i, d)) \times \prod_{s \neq i,j} \mu_s(A_s^{**}(s)) \right\}. \tag{9}$$

Using (8) and the facts that

$$\textstyle\sum_{r=0}^{n-1} \mu_j(B_j^n(r, i, d)) \leqq \mu_j(A_j^{**}(j)) \quad \text{and} \quad \mu_j(A_j^{**}(j)) \leqq 1 \quad \text{for } \forall j,$$

we can transform (9) to $\boldsymbol{\mu}(A^{**}(i)) \leqq D(i)N\varepsilon$. Since ε is arbitrary, we conclude that $\boldsymbol{\mu}(A^{**}(i)) = 0$.

Next suppose that for $\bar{a}_i \in A_i$, $\mu_j(f_{ij}^d(\bar{a}_i)) = 0$ for all $j \neq i$ and $1 \leqq d \leqq D(i)$. We then have

$$\begin{aligned} \boldsymbol{\mu}_{-i}(A_{-i}^*(\bar{a}_i)) &\leqq \textstyle\sum_{j \neq i} \sum_{1 \leqq d \leqq D(i)} \boldsymbol{\mu}_{-i}(\{\boldsymbol{a}_{-i} \in A_{-i} | a_j = f_{ij}^d(\bar{a}_i)\}) \\ &\leqq \textstyle\sum_{j \neq i} \sum_{i \leqq d \leqq D(i)} \mu_j(\{f_{ij}^d(\bar{a}_i)\}) \prod_{s \neq i,j} \mu_s(A_s) = 0, \end{aligned}$$

as desired. ||

Lemma 2. *Let $A_i \subseteq R^1$, $(i = 1, \ldots, N)$ be a closed interval and let $U_i: A \to R^1$, $(i = 1, \ldots, N)$, be continuous, except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by (2). Suppose $\forall i$, U_i is bounded. Let A_i^n be a finite subset of A_i with the property that*

$$\sup_{a_i \in A_i} \inf_{a_i^n \in A_i^n} |a_i - a_i^n| \leqq 1/n.$$

Let $(\mu_1^n, \ldots, \mu_N^n)$ be an equilibrium vector of mixed strategies for the finite game $[(A_i^n, U_i); i = 1, \ldots, N]$, and let $(\mu_1^*, \ldots, \mu_N^*) = \lim_{n \to \infty} (\mu_1^n, \ldots, \mu_N^n)$.[12] *If for some i, $\exists \bar{a}_i \in A_i$ such that*

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i}) d\boldsymbol{\mu}_{-i}^* > \lim_{n \to \infty} \int_A U_i(\boldsymbol{a}) d\boldsymbol{\mu}^n + \varepsilon \quad \textit{for some } \varepsilon > 0, \tag{10}$$

then $\boldsymbol{\mu}_{-i}^*(\bar{A}_{-i}^{**}(\bar{a}_i)) > 0$, where $\bar{A}_{-i}^{**}(\bar{a}_i)$ is the closure of the set

$$A_{-i}^{**}(\bar{a}_i) \equiv \{\boldsymbol{a}_{-i} \in A_{-i} | (\bar{a}_i, \boldsymbol{a}_{-i}) \in A^{**}(i)\}.$$[13]

Remark. From Lemma 2 we may conclude that $\forall \bar{a}_i \notin A_i^{**}(i)$,

$$\int U_i(\bar{a}_i, \boldsymbol{a}_{-i}) d\boldsymbol{\mu}_{-i} \leqq \lim_{n \to \infty} \int U_i(\boldsymbol{a}) d\boldsymbol{\mu}^n.$$

Proof. Suppose that the hypotheses of the Lemma are satisfied, but that $\mu^*_{-i}(\bar{A}^{**}_{-i}(\bar{a}_i))=0$. Because $\bar{A}^{**}_{-i}(\bar{a}_i)$ is closed, for each $\varepsilon>0$, $\exists$ an open set $A^{**}_{-i}\supseteq \bar{A}^{**}_{-i}(\bar{a}_i)$ such that

$$\mu^*_{-i}(\bar{A}^{**}_{-i})<\varepsilon/4M, \tag{11}$$

where $M>\sup|U_i|$, and $\bar{A}^{**}_{-i}$ is the closure of A^{**}_{-i}.

Using (10) and (11) we may obtain

$$\int_{\tilde{A}_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})\,d\mu^*_{-i}>\lim_{n\to\infty}\int_A U_i(\boldsymbol{a})\,d\mu^n+3\varepsilon/4 \tag{12}$$

where $\tilde{A}_{-i}\equiv A_{-i}-A^{**}_{-i}$.

Choose an open neighbourhood $B(\bar{a}_i)$ of $\bar{a}_i$, such that if $a_i\in B(\bar{a}_i)$ then $A^{**}_{-i}(a_i)\subseteq A^{**}_{-i}$. Then U_i is continuous when restricted to $B(\bar{a}_i)\times\tilde{A}_{-i}$. From (11) and (12) we conclude that $\exists t>0$ and $\bar{a}^t_i\in A^t_i\cap B(\bar{a}_i)$ such that

$$\int_{\tilde{A}_{-i}} U_i(\bar{a}^t_i, \boldsymbol{a}_{-i})\,d\mu^*_{-i}>\lim_{n\to\infty}\int_A U_i(\boldsymbol{a})\,d\mu^n+3\varepsilon/4. \tag{13}$$

Because U_i is continuous on $\{\bar{a}^t_i\}\times\tilde{A}_{-i}$ we can invoke the Tietz extension lemma to continuously extend U_i as a function of $\boldsymbol{a}_{-i}$ to $\{\bar{a}^t_i\}\times A_{-i}$. If $\tilde{U}_i$ is the extension, then we may assume that $\sup|\tilde{U}_i|<M$.

From (11) and (13) we have

$$\int_{A_{-i}} \tilde{U}_i(\bar{a}^t_i, \boldsymbol{a}_{-i})\,d\mu^*_{-i}>\lim_{n\to\infty}\int_A U_i(\boldsymbol{a})\,d\mu^n+\varepsilon/2. \tag{14}$$

But from the hypotheses, inequalities (12) and (14) and the continuity of $\tilde{U}_i$, we may conclude that $\exists\bar{n}>0$ such that for $n>\bar{n}$,

$$\int_{A_{-i}} \tilde{U}_i(\bar{a}^t_i, \boldsymbol{a}_{-i})\,d\mu^n_{-i}>\int_A U_i(\boldsymbol{a})\,d\mu^n+\varepsilon/2 \tag{15}$$

and from (11) that

$$\mu^n_{-i}(\bar{A}^{**}_{-i})<\varepsilon/4M. \tag{16}$$

Therefore, from (15) and (16) we obtain, for $n>\bar{n}$,

$$\int_{A_{-i}} U_i(\bar{a}^t_i, \boldsymbol{a}_{-i})\,d\mu^n_{-i}>\int_A U_i(\boldsymbol{a})\,d\mu^n+\varepsilon/4. \tag{17}$$

But for $n>t$, (17) contradicts the hypothesis that $\mu^n=(\mu^n_1,\dots,\mu^n_N)$ constitutes an equilibrium for the game $[(A^n_i, U_i); i=1,\dots,N]$. Thus $\mu^*_{-i}(\bar{A}^{**}_{-i}(\bar{a}_i))>0$. ‖

We may now use Lemmas 1 and 2 to state and prove:

Theorem 4. *Let $A_i\subseteq R^1$, $(i=1,\dots,N)$, be a closed interval, and let $U_i: A\to R^1$, $(i=1,\dots,N)$, be continuous, except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by* (2).

Suppose $\forall i$, U_i is bounded. Let A^n_i be a finite subset of A_i with the property that

$$\sup_{a_i\in A_i}\inf_{a^n_i\in A^n_i}|a_i-a^n_i|<1/n.$$

Let $(\mu_1^n, \ldots, \mu_N^n)$ *be an equilibrium vector of mixed strategies for the finite game* $[(A_i^n, U_i); i=1, \ldots, N]$, *and let* $(\mu_1^*, \ldots, \mu_N^*) = \lim_{n\to\infty}(\mu_1^n, \ldots, \mu_N^n)$; *(see footnote 12). Suppose,* $\forall ij, d$ *and* $\forall \bar{a}_i \in \bar{A}_i^{**}(i)$, $\mu_j^*(\{f_{ij}^d(\bar{a}_i)\}) = 0$. *Then* $(\mu_1^*, \ldots, \mu_N^*) \equiv \boldsymbol{\mu}^*$ *constitutes an equilibrium for the infinite game* $[(A_i, U_i); i=1, \ldots, N]$.

Proof. Suppose that the hypotheses of the theorem are satisfied, but that $(\mu_1^*, \ldots, \mu_N^*)$ is not an equilibrium. Then some agent, say agent i, can do better by playing some strategy other than μ_i^*. That is, $\exists \bar{a}_i \in A_i$, and $\varepsilon > 0$ such that

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}_{-i}^* > \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^* + \varepsilon. \tag{18}$$

Obviously, $\{\bar{a}_i\} \times A_{-i}^{**}(\bar{a}_i) \subseteq A^{**}(i)$.

Therefore, by hypothesis and Lemma 1, we have

$$\boldsymbol{\mu}_{-i}^*(A_{-i}^{**}(\bar{a}_i)) = 0, \tag{19}$$

and

$$\boldsymbol{\mu}^*(A^{**}(i)) = 0. \tag{20}$$

Since U_i is continuous on $A - A^{**}(i)$, we may conclude from (20) that

$$\lim_{n\to\infty} \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n = \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^*.$$

Thus, from (18) we have

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}_{-i}^* > \lim_{n\to\infty} \int_A U_i(a_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}^n + \varepsilon. \tag{21}$$

But (19) and (21) together contradict the conclusion of Lemma 2. Therefore $\boldsymbol{\mu}^* = (\mu_1^*, \ldots, \mu_N^*)$ is an equilibrium for the game $[(A_i, U_i); i=1, \ldots, N]$. $\|$

Remark. If $\forall i$, $A^{**}(i)$ is empty, (i.e. U_i is continuous on A), Theorem 4 asserts that the game $[(A_i, U_i); i=1, \ldots, N]$ possesses a mixed-strategy equilibrium. This is Theorem 3 above.

Remark. We have noted that in Example 1, where $N=2$ and $A_1 = A_2 = [0, 1]$, we may take $D(1) = D(2) = 2$, $f_{12}^1(a_1) = a_1$ and $f_{12}^2(a_1) = a_1 + \frac{1}{2}$. We also noted that for this game $A^*(i) = A^{**}(i)$, $(i=1, 2)$, and therefore, that $\bar{A}_1^{**}(1) = \bar{A}_2^{**}(2) = [0, 1]$. Theorem 4 implies that the limits of a sequence of equilibria of successively finer finite approximations of Example 1 must possess atoms.

4.4. *The main existence theorem*

In this section we state and prove our main result (Theorem 5). First, we prove two lemmas which will be useful. In Section 4.2 we presented a verbal account of them and their motivation.

Lemma 3. *Let* $A_i \subseteq R^1$, $(i=1, \ldots, N)$ *be a closed interval, and let* $U_i: A \to R^1$, $(i = 1, \ldots, N)$, *be continuous except on a subset* $A^{**}(i)$ *of* $A^*(i)$, *where* $A^*(i)$ *is defined by* (2). *Suppose* $\forall i$, U_i *is bounded. Let* A_i^n *be a finite subset of* A_i *with the property that*

$$\sup_{a_i \in A_i} \inf_{a_i^n \in A_i^n} |a_i - a_i^n| < 1/n.$$

Let $(\mu_1^n, \dots, \mu_N^n)$ *be an equilibrium vector of mixed strategies for the finite game* $[(A_i^n, U_i); i = 1, \dots, N]$, *and let* $(\mu_1^*, \dots, \mu_N^*) = \lim_{n\to\infty}(\mu_1^n, \dots, \mu_N^n)$; (*see footnote* 12). *Then,* $\forall i$

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}_{-i}^* \leqq \lim_{n\to\infty} \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n$$

for all but countably many $\bar{a}_i \in A_i$.

Proof. Suppose that the hypotheses of the lemma hold but that $\exists i$ such that

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}_{-i}^* > \lim_{n\to\infty} \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n \tag{22}$$

for uncountably many $\bar{a}_i \in A_i$.

By Lemma 2 we conclude that for each $\bar{a}_i \in A_i$ which satisfies (22), $\boldsymbol{\mu}_{-i}^*(\bar{A}_{-i}^{**}(\bar{a}_i)) > 0$. But then, by Lemma 1 we know that for each such $\bar{a}_i$, $\exists j$ $(j \neq i)$ and $\exists d$ with $1 \leqq d \leqq D(i)$ such that

$$\mu_j^*(\{f_{ij}^d(\bar{a}_i)\}) > 0. \tag{23}$$

Since we are supposing here that the number of $\bar{a}_i$'s satisfying (22) is uncountable, there must exist j and d such that (23) holds for uncountably many $\bar{a}_i$'s. But f_{ij}^d is one-to-one, so that we may conclude that μ_j^* has uncountably many atoms. But this is an impossibility. Thus (22) can hold for only countably many $\bar{a}_i$'s. ||

The hypotheses of Lemma 3 are satisfied by Example 1. The economic games that we study in the sequel are (upper semi) continuous-sum games. Moreover, Example 1, being a zero-sum game is, of course, an upper semi continuous-sum game. We now come to a property of payoff functions, however, that is satisfied by the games in the sequel, but *not* by Example 1.

Definition 6. $U_i(a_i, \boldsymbol{a}_{-i})$ *is weakly lower semi-continuous* in a_i if $\forall \bar{a}_i \in A_i^{**}(i)$, $\exists \lambda \in [0, 1]$ such that $\forall \boldsymbol{a}_{-i} \in A_{-i}^{**}(\bar{a}_i)$, $\lambda \liminf_{a_i \to^{-} \bar{a}_i} U_i(a_i, \boldsymbol{a}_{-i}) + (1-\lambda) \liminf_{a_i \to^{+} \bar{a}_i} U_i(a_i, \boldsymbol{a}_{-i}) \geqq U_i(\bar{a}_i, \boldsymbol{a}_{-i})$.[14]

(If $\bar{a}_i$ is the right end point of A_i and, therefore, $\liminf_{a_i \to^{+} \bar{a}_i} U_i(a_i, \boldsymbol{a}_{-i})$ is not defined, the definition reduces to the condition $\liminf_{a_i \to^{-} \bar{a}_i} U_i(a_i, \boldsymbol{a}_{-i}) \geqq U_i(\bar{a}_i, \boldsymbol{a}_{-i})$; i.e. that $U_i(a_i, \boldsymbol{a}_{-i})$ is left lower semi-continuous in a_i at $\bar{a}_i$. Similarly, if $\bar{a}_i$ is the left end point of A_i, then the definition reduces to the condition $\liminf_{a_i \to^{+} \bar{a}_i} U_i(a_i, \boldsymbol{a}_{-i}) \geqq U_i(\bar{a}_i, \boldsymbol{a}_{-i})$; i.e. that $U_i(a_i, \boldsymbol{a}_{-i})$ is right lower semi-continuous in a_i at $\bar{a}_i$.)

Remark. Notice that in Example 1 $U_1(a_1, a_2)$ is not weakly lower semi-continuous in a_1 at the point $(1, 1)$.

Lemma 4. *Let* $A_i \subseteq R^1$, $(i = 1, \dots, N)$, *be a closed interval, and let* $U_i : A \to R^1$, $(i = 1, \dots, N)$, *be continuous, except on a subset* $A^{**}(i)$ *of* $A^*(i)$, *where* $A^*(i)$ *is defined by* (2). *Suppose* $\sum_{i=1}^N U_i(\boldsymbol{a})$ *is upper semi-continuous,* $\forall i$ $U_i(a_i, \boldsymbol{a}_{-i})$ *is bounded, and is weakly lower semi-continuous in* a_i. *Let* A_i^n *be a finite subset of* A_i *with the property that*

$$\sup_{a_i \in A_i} \inf_{a_i^n \in A_i^n} |a_i - a_i^n| < 1/n.$$

Let $(\mu_1^n, \ldots, \mu_N^n)$ be an equilibrium vector of mixed strategies for the finite game $[(A_i^n, U_i); i=1, \ldots, N]$, and let $(\mu_1^*, \ldots, \mu_N^*) = \lim_{n\to\infty}(\mu_1^n, \ldots, \mu_N^n)$; (see footnote 12). Then, $\forall i$.

$$\lim_{n\to\infty} \int_A U_i(\boldsymbol{a})d\boldsymbol{\mu}^n = \int_A U_i(\boldsymbol{a})d\boldsymbol{\mu}^*. \tag{24}$$

Proof. Suppose that the hypotheses of the Lemma hold. Then $\forall i$ there exists a subsequence of $\{\int U_i(\boldsymbol{a})d\boldsymbol{\mu}^n\}$ which converges. Thus without loss of generality we may assume that $\forall i$, $\lim_{n\to\infty}\int_A U_i(\boldsymbol{a})d\boldsymbol{\mu}^n$ exists. Now suppose $\exists i$ such that (24) is not satisfied. Then $\exists \varepsilon > 0$ such that either

$$\lim_{n\to\infty} \int U_i(\boldsymbol{a})d\boldsymbol{\mu}^n \geqq \int U_i(\boldsymbol{a})d\boldsymbol{\mu}^* + \varepsilon, \tag{25}$$

or

$$\lim_{n\to\infty} \int U_i(\boldsymbol{a})d\boldsymbol{\mu}^n \leqq \int U_i(\boldsymbol{a})d\boldsymbol{\mu}^* - \varepsilon. \tag{26}$$

Suppose (25) holds for this i. Then since $\sum_{i=1}^N U_i(\boldsymbol{a})$ is upper semi-continuous we may conclude that

$$\int \textstyle\sum_{i=1}^N U_i(\boldsymbol{a})d\boldsymbol{\mu}^* \geqq \lim_{n\to\infty} \displaystyle\int \textstyle\sum_{i=1}^N U_i(\boldsymbol{a})d\boldsymbol{\mu}^n.$$

It follows therefore that $\exists j \neq i$ such that (26) holds for agent j. We conclude therefore that if (24) does not hold for all agents then there exists at least one agent for whom (26) holds. For this agent, j, $\exists \bar{a}_j \in A_j$ such that

$$\int_{A_{-j}} U_j(\bar{a}_j, \boldsymbol{a}_{-j})d\boldsymbol{\mu}_{-j}^* \geqq \lim_{n\to\infty} \int U_j(\boldsymbol{a})d\boldsymbol{\mu}^n + \varepsilon. \tag{27}$$

Now suppose U_j is weakly lower semi-continuous in a_j. Then it follows that $\int_{A_{-j}} U_j(a_j, \boldsymbol{a}_{-j})d\boldsymbol{\mu}_{-j}^*$ is either right or left lower semi-continuous as a function of a_j at $\bar{a}_j$. Without loss of generality assume that it is right lower semi-continuous. Then it follows from (27) that

$$\liminf_{a_j \overset{+}{\to} \bar{a}_j} \int_{A_{-j}} U_j(a_j, \boldsymbol{a}_{-j})d\boldsymbol{\mu}_{-j}^* \geqq \int_{A_{-j}} U_j(\bar{a}_j, \boldsymbol{a}_{-j})d\boldsymbol{\mu}_j^* \geqq \lim_{n\to\infty} \int U_j(\boldsymbol{a})d\boldsymbol{\mu}^n + \varepsilon. \tag{28}$$

It follows that $\exists \delta > 0$ such that $\forall \tilde{a}_j \in [\bar{a}_j, \bar{a}_j + \delta]$,

$$\int_{A_{-j}} U_j(\tilde{a}_j, \boldsymbol{a}_{-j})d\boldsymbol{\mu}_{-j}^* > \lim_{n\to\infty} \int U_j(\boldsymbol{a})d\boldsymbol{\mu}^n.$$

But this contradicts Lemma 3. Therefore (24) must hold for all agents. ||

Theorem 5. *Let $A_i \subseteq R^1$ $(i=1, \ldots, N)$ be a closed interval and let $U_i: A \to R^1$ $(i=1, \ldots, N)$ be continuous except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by (2). Suppose $\sum_{i=1}^N U_i(\boldsymbol{a})$ is upper semi-continuous and $U_i(a_i, \boldsymbol{a}_{-i})$ is bounded and weakly lower semi-continuous in a_i. Then the game $[(A_i, U_i); i=1, \ldots, N]$ possesses a mixed-strategy equilibrium.*

Proof. Suppose the hypotheses of the theorem are satisfied. $\forall i$, and $n>0$, let A_i^n be a finite subset of A_i with the property that

$$\sup_{a_i \in A_i} \inf_{a_i^n \in A_i^n} |a_i - a_i^n| < 1/n.$$

For each $n>0$, let $(\mu_1^n, \ldots, \mu_N^n)$ be a mixed-strategy equilibrium of the game $[(A_i^n, U_i); i=1, \ldots, N]$. Let $(\mu_1^*, \ldots, \mu_N^*) = \lim_{n\to\infty} (\mu_1^n, \ldots, \mu_N^n)$; (see footnote 12). If $\boldsymbol{\mu}^*$ is not an equilibrium of the game $[(A_i, U_i); i=1, \ldots, N]$, then $\exists i$ and $\exists \bar{a}_i \in A_i$ such that

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}_{-i}^* > \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^*. \tag{29}$$

From Lemma 4, (29) implies

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}_{-i}^* > \lim_{n\to\infty} \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n. \tag{30}$$

Now we can use an argument identical to the one used in Lemma 4 to show that if (30) holds then there is an uncountable set of $\tilde{a}_i$'s, with $\tilde{a}_i \in A_i$ such that

$$\int U_i(\tilde{a}_i, \boldsymbol{a}_{-i})\, d\boldsymbol{\mu}_{-i}^* > \lim_{n\to\infty} \int U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n,$$

and this contradicts Lemma 3. Therefore $\boldsymbol{\mu}^*$ is an equilibrium of the game $[(A_i, U_i); i=1, \ldots, N]$. ||

Our sets $A^*(i)$ rule out discontinuities in agent i's payoff that occur independently of other agents' payoffs. This is limiting. For suppose that a quantity-setting firm must bear a set-up cost to produce any positive output level. Then its profit will be discontinuous at zero output regardless of what the other firms do.

We wish to modify Theorem 5 to allow for this sort of discontinuity. This is achieved in

Theorem 5a. *Suppose that the game $[(A_i, U_i); i=1, \ldots, N]$ satisfies the hypotheses of Theorem 5, except that for some i there exists a strategy $\hat{a}_i \in A_i$ such that for all $\bar{\boldsymbol{a}}_{-i} \in A_{-i}$,*

(i) $\lim_{a_i \bar{\to} \hat{a}_i, \boldsymbol{a}_{-i} \to \bar{\boldsymbol{a}}_{-i}} U_i(a_i, \boldsymbol{a}_{-i})$ *exists and equals* $U(\hat{a}_i, \bar{\boldsymbol{a}}_{-i})$

(ii) $\lim_{a_i \stackrel{+}{\to} \hat{a}_i} U_i(a_i; \bar{\boldsymbol{a}}_{-i})$ *exists, is less than or equal to* $U(\hat{a}_i, \bar{\boldsymbol{a}}_{-i})$ *and is continuous in* $\bar{\boldsymbol{a}}_{-i}$. *Then the game has a mixed-strategy equilibrium.*

Proof. The proof consists of enlarging player i's strategy space, extending the U_j's so that the game satisfies the hypotheses of Theorem 5 on the enlarged space, and then showing that an equilibrium of the modified game corresponds to an equilibrium of the original game.

Let $A_i = [a_i^*, a_i^{**}]$ and take $\hat{A}_i = [a_i^*, a_i^{**}+1]$. Define $\hat{U}_j : \hat{A}_i \times A_{-i} \to R^1, j=1, \ldots, N$, so that

$$\hat{U}_i(a_i, \boldsymbol{a}_{-i}) = \begin{cases} U_i(a_i, \boldsymbol{a}_{-i}), & \text{if } a_i \leqq \hat{a}_i \\ (a_i - \hat{a}_i) \lim_{\tilde{a}_i \stackrel{+}{\to} \hat{a}_i} U_i(\tilde{a}_i, \boldsymbol{a}_{-i}) + (1 - a_i + \hat{a}_i) U_i(\hat{a}_i, \boldsymbol{a}_{-i}) & \\ \quad \text{if } \hat{a}_i < a_i \leqq \hat{a}_i + 1, & \\ U_i(a_i - 1, \boldsymbol{a}_{-i}), & \text{if } a_i > \hat{a}_i + 1 \end{cases} \tag{31}$$

and for $j \neq i$

$$\hat{U}_j(a_i, \boldsymbol{a}_{-i}) = \begin{cases} U_j(a_i, \boldsymbol{a}_{-i}), & \text{if } a_i \leqq \hat{a}_i \\ U_j(\hat{a}_i, \boldsymbol{a}_{-i}), & \text{if } \hat{a}_i < a_i \leqq \hat{a}_i + 1 \\ U_j(a_i - 1, \boldsymbol{a}_{-i}), & \text{if } a_i > \hat{a}_i + 1. \end{cases} \tag{32}$$

Because $\lim_{\tilde{a}_i \xrightarrow{+} \hat{a}_i} U_i(\tilde{a}_i, \boldsymbol{a}_{-i})$ exists, this modified game is well-defined. Furthermore, because $(a_i - \hat{a}_i) \lim_{\tilde{a}_i \xrightarrow{+} \hat{a}_i} U_i(\tilde{a}_i, \boldsymbol{a}_{-i}) + (1 - a_i + \hat{a}_i) U_i(\hat{a}_i, \boldsymbol{a}_{-i})$ is continuous at any point where $\hat{a}_i \leqq a_i \leqq \hat{a}_i + 1$ and converges to $U_i(\hat{a}_i, \bar{\boldsymbol{a}}_{-i})$ as $(a_i, \boldsymbol{a}_{-i})$ tends to $(\hat{a}_i, \bar{\boldsymbol{a}}_{-i})$, $\hat{U}_i$ is continuous at all points $(a_i, \boldsymbol{a}_{-i})$, where $\hat{a}_i \leqq a_i \leqq \hat{a}_i + 1$. Notice too that (32) introduces no new discontinuities in the payoff functions of players other than i. Hence the modified game satisfies the hypotheses of Theorem 5. We conclude that there exists an equilibrium $(\hat{\mu}_1, \ldots, \hat{\mu}_N)$. For any set $E \subseteq A_i$, let

$$\mu_i^*(E) = \begin{cases} \hat{\mu}_i(E \cap [a_i^*, \hat{a}_i)) + \hat{\mu}_i([\hat{a}_i, \hat{a}_i + 1]) \\ \quad + \hat{\mu}_i(\{a_i | a_i - 1 \in E \text{ and } a_i - 1 > \hat{a}_i\}) \quad \text{if } \hat{a}_i \in E \\ \hat{\mu}_i(E \cap [a_i^*, \hat{a}_i)) + \hat{\mu}_i(\{a_i | a_i - 1 \in E \text{ and } a_i - 1 > \hat{a}_i\}) \quad \text{if } \hat{a}_i \notin E. \end{cases}$$

Take $\mu_j^* = \hat{\mu}_j$ for all $j \neq i$. Notice that μ_i^* shifts all the probability mass of $\hat{\mu}_i$ in $[\hat{a}_i, \hat{a}_i + 1]$ to $\hat{a}_i$. Thus because $U_i(\hat{a}_i, \boldsymbol{a}_{-i}) \geqq \lim_{\tilde{a}_i \xrightarrow{+} \hat{a}_i} U_i(a_i, \boldsymbol{a}_{-i})$ we see from (31) that μ_i^* must be an optimal strategy for i in the original game given that $\hat{\mu}_i$ is optimal in the modified game. Because this shift in mass does not affect the payoffs of agents other than i, $\hat{\mu}_j$ is optimal in the original game for each $j \neq i$. Hence $(\mu_1^*, \ldots, \mu_N^*)$ is an equilibrium for the original game. ||

Remark. Theorem 5a implies that an equilibrium exists in the Cournot model with set-up costs studied by Novshek (1980) and others, even when set-up costs are large.

Theorem 5a generalised Theorem 5 in one direction—roughly speaking, that the f_{ij}^d's are not restricted to be one-to-one mappings. In Theorem 5b below we generalise Theorem 5 in another direction. Notice that, in the proof of Theorem 5, the hypothesis that the sum of agents' payoff functions is upper semicontinuous is invoked only to show that if $\{\mu^n\}$ is a sequence of vectors of measures converging to $\boldsymbol{\mu}^*$, there exists an agent i for whom (25) holds. We can guarantee the existence of such a player, however, without requiring upper semicontinuity. What is needed is that at any point where one player's payoff falls, another's rises. To formalise this idea we restrict attention to the case of two agents and of a discontinuity set consisting of the main diagonal.

Theorem 5b. *Suppose that* $N = 2$ *and that* $A_1 = A_2 = [a^*, a^{**}]$. *For* $i = 1, 2$ *suppose that* $U_i: A \to R^1$ *is bounded and continuous except on the subset* $\{(a_1, a_2) | a_1 = a_2\}$. *For each* $a \in [a^*, a^{**}]$ *assume that there exists* $i \in \{1, 2\}$ *such that*

(i) $\lim_{a_1 \xrightarrow{-} a, a_2 \xrightarrow{+} a} U_i(a_1, a_2) \geqq U_i(a, a) \geqq \lim_{a_1 \xrightarrow{+} a, a_2 \xrightarrow{-} a} U_i(a_1, a_2)$

and

(ii) $\lim_{a_1 \xrightarrow{-} a, a_2 \xrightarrow{+} a} U_j(a_1, a_2) \leqq U_j(a, a) \leqq \lim_{a_1 \xrightarrow{+} a, a_2 \xrightarrow{-} a} U_j(a_1, a_2), j \neq i,$

where the left (right) inequality in (i) *is strict if and only if the right (left) inequality in* (ii) *strict. Then the game* $[(A_i, U_i); i = 1, 2]$ *has a mixed strategy equilibrium.*

Proof. For each $a \in [a^*, a^{**}]$ choose $\bar{U}_i(a)$ and $\bar{U}_j(a)$ and define

$$\hat{U}_i(a, a) = \begin{bmatrix} U_i(a, a), & \text{if the left inequality in (i) holds with equality} \\ \bar{u}_i(a), & \text{if the left inequality in (i) holds strictly} \end{bmatrix} \tag{33}$$

and

$$\hat{U}_j(a, a) = \begin{bmatrix} U_j(a, a), & \text{if the right inequality in (ii) holds with equality} \\ \bar{u}_j(a), & \text{if the right inequality in (ii) holds strictly} \end{bmatrix} \tag{34}$$

where

$$\lim_{a_1 \bar{\to} a, a_2 \overset{+}{\to} a} U_i(a_1, a_2) > \bar{u}_i(a) > U_i(a, a), \tag{35}$$

$$\lim_{a_1 \overset{+}{\to} a, a_2 \bar{\to} a} U_j(a_1, a_2) > \bar{u}_j(a) > U_j(a, a), \tag{36}$$

and

$$\bar{u}_i(a) + \bar{u}_j(a) \geqq \begin{cases} \lim_{a_1 \bar{\to} a, a_2 \overset{+}{\to} a} (U_i(a_1, a_2) + U_j(a_1, a_2)) \\ \lim_{a_1 \overset{+}{\to} a, a_2 \bar{\to} a} (U_i(a_1, a_2) + U_j(a_1, a_2)) \end{cases}. \tag{37}$$

For $a_1 \neq a_2$ take

$$\hat{U}_i(a_1, a_2) = U_i(a_1, a_2) \tag{38}$$

and

$$\hat{U}_j(a_1, a_2) = U_j(a_1, a_2). \tag{39}$$

Because U_i and U_j are continuous where $a_1 \neq a_2$, (38) and (39) imply that $\hat{U}_1 + \hat{U}_2$ is also continuous there. From (33), (34) and (37), $\hat{U}_1 + \hat{U}_2$ is upper semicontinuous at points where $a_1 = a_2$. From (33) and (35), $\hat{U}_i$ is either left or right lower semicontinuous at (a, a) and, from (34) and (36), so is $\hat{U}_j$. Hence, the game $[(A_i, \hat{U}_i); i = 1, 2]$ has a mixed strategy equilibrium $(\hat{\mu}_1, \hat{\mu}_2)$. We will now show that $(\hat{\mu}_1, \hat{\mu}_2)$ is an equilibrium for the original game.

Choose $\hat{a}_1 \in \operatorname{supp} \hat{\mu}_1$. Then,

$$\int \hat{U}_1(\hat{a}_1, a_2) d\hat{\mu}_2 \geqq \int \hat{U}_1(a_1, a_2) d\hat{\mu}_2 \quad \text{for all } a_1 \in A_1. \tag{40}$$

If $\hat{\mu}_2(\hat{a}_1) > 0$ and if U_1 is discontinuous at $(\hat{a}_1, \hat{a}_1)$, then from (33) and (34), there exists a_1' close to $\hat{a}_1$ such that $\int \hat{U}_1(a_1', a_2) d\hat{\mu}_2 > \int \hat{U}_1(a_1, a_2) d\mu_2$, a contradiction of (40). Hence

$$\int \hat{U}_1(\hat{a}_1, a_2) d\hat{\mu}_2 = \int U_1(\hat{a}_1, a_2) d\hat{\mu}_2. \tag{41}$$

But from (33)-(36), $\hat{U}_1(a_1, a_2) \geqq U_1(a_1, a_2)$ for all (a_1, a_2). Therefore,

$$\int \hat{U}_1(a_1, a_2) d\hat{\mu}_2 \geqq \int U_1(a_1, a_2) d\hat{\mu}_2. \tag{42}$$

But (41) and (42) imply that

$$\int U_1(\hat{a}_1, a_2) d\hat{\mu}_2 \geqq \int U_1(a_1, a_2) d\hat{\mu}_2,$$

i.e., $\hat{\mu}_1$ is best response to $\hat{\mu}_2$ in the original game. Similarly $\hat{\mu}_2$ is a best response to $\hat{\mu}_1$. ||

5. SYMMETRIC GAMES

Definition 7. Let $\bar{\bar{A}} \subseteq R^m (m \geqq 1)$ be non-empty and compact, and let

$$A = \underbrace{\bar{\bar{A}} \times \cdots \times \bar{\bar{A}}}_{N \text{ times}}.$$

For $U_i: A \to R^1, i = 1, \ldots, N, [(\bar{\bar{A}}, U_i); i = 1, \ldots, N]$ is a symmetric game if for all permutations, π, on the set $\{1, \ldots, N\}$, all $\boldsymbol{a} \in A$ and all i,

$$U_i(a_1, \ldots, a_i, \ldots, a_N) = U_{\pi(i)}(a_{\pi(1)}, \ldots, a_{\pi(i)}, \ldots, a_{\pi(N)}).$$

The central result of this section, Theorem 6, states that under suitable assumptions mixed-strategy equilibria of discontinuous symmetric games are atomless on the set of discontinuities.

Definition 8. An equilibrium in mixed strategies, $(\mu_1^*, \ldots, \mu_i^*, \ldots, \mu_N^*)$, is symmetric if μ_i^* is independent of i.

Lemma 5 (Fan). *Let $X \subseteq R^n$, $(n \geqq 1)$, be non-empty, convex and compact. Let $F: X \times X \to R^1$ be continuous, and concave in its first argument, and suppose that $F(x, x) = 0$ for $x \in X$. Then $\exists x^* \in X$ such that* $\max_{y \in X} F(y, x^*) = 0$.

Proof. $\forall y \in X$, let $\phi(y) \equiv \{x \in X | F(x, y) = \max_{x' \in X} F(x', y)\}$. It is immediate that ϕ is non-empty, convex- and compact-valued and upper hemi-continuous. Therefore, by Kakutani fixed point theorem, $\exists y^* \in X$ such that $y^* \in \phi(y^*)$; that is,

$$\max_{x \in X} F(x, y^*) = F(y^*, y^*) = 0. \quad \|$$

We can now establish a simple result on symmetric, finite games.

Lemma 6. *Let $[(\bar{\bar{A}}, U_i); i = 1, \ldots, N]$ be a symmetric game, where $\bar{\bar{A}} \subseteq R^m (m \geqq 1)$, is non-empty and finite. Then the game possesses a symmetric mixed-strategy equilibrium.*[15]

Proof. Let $D(\bar{\bar{A}})$ be the set of all (Borel) probability measures on $\bar{\bar{A}}$. For $\mu, \nu \in D(\bar{\bar{A}})$ define

$$F(\mu, \nu) \equiv \int U_1(a_1, \ldots, a_N) d(\mu(a_1) \times \nu(a_2) \times \cdots \times \nu(a_N))$$
$$- \int U_1(a_1, \ldots, a_N) d(\nu(a_1) \times \nu(a_2) \times \cdots \times \nu(a_N)).$$

Clearly, F is continuous in μ and ν and linear (and therefore concave) in μ. Moreover, $F(\mu, \mu) = 0$. Therefore F satisfies the hypotheses of Lemma 5. Thus $\exists \mu^* \in D(\bar{\bar{A}})$ such that $\max_{\mu \in D(\bar{\bar{A}})} F(\mu, \mu^*) = F(\mu^*, \mu^*) = 0$. It follows that μ^* is the best response for agent 1 if each of the remaining agents uses μ^*. By symmetry, μ^* is the best response for any agent i if each of the remaining agents μ^*. Thus $(\mu^*, \ldots, \mu^*)$ is an equilibrium. $\|$

We can now state

Lemma 7. *Let* $[(\bar{\bar{A}}, U_i); i=1, \ldots, N]$ *be a symmetric game, where* $\bar{\bar{A}} \subseteq R^1$ *is non-empty and compact, and where* U_i *satisfies the conditions of Theorem* 5. *Then the game possesses a symmetric mixed strategy equilibrium.*

Proof. Use the proof of Theorem 5 and Lemma 6. ||

Theorem 6 below presents a set of conditions which ensures the existence of equilibrium mixed strategies that are atomless on discontinuity sets. The computation of equilibrium strategies is particularly simplified when this is so. The examples analysed by Karlin (1959), Beckmann (1965), Rosenthal (1980) and Varian (1980) satisfy the postulates of Theorem 6. The theorem, therefore, also provides an explanation of why the computational techniques these authors rely on actually work for their models.

We are now concerned with symmetric games $[(\bar{\bar{A}}, U_i); i=1, \ldots, N]$, with $\bar{\bar{A}} \subseteq R^1$, possessing the property that $\forall \bar{a}_i \in A_i^{**}(i)$, the corresponding point on the "diagonal",

$$\underbrace{(\bar{a}_i, \bar{a}_i, \ldots, \bar{a}_i)}_{N\text{-times}},$$

is a point of discontinuity of U_i. As we shall see in the sequel, several well-known economic games share this property. What we require is that the inequality in the definition of weak lower semi-continuity (Definition 6) be strict on the diagonal. We shall call this stronger condition property (α).

Property (α). $\forall \bar{a}_i \in A_i^{**}(i), \exists \lambda \in [0, 1]$ such that for all $\boldsymbol{a}_{-i} \in A_{-i}^{**}(\bar{a}_i)$

$$\lambda \liminf_{a_i \bar{\to} \bar{a}_i} U_i(a_i, \boldsymbol{a}_{-i}) + (1-\lambda) \liminf_{a_i \underset{+}{\to} \bar{a}_i} U_i(a_i, \boldsymbol{a}_{-i}) \geqq U_i(\bar{a}_i, \boldsymbol{a}_{-i}),$$

where the inequality is strict if

$$\boldsymbol{a}_{-i} = \underbrace{(\bar{a}_i, \ldots, \bar{a}_i)}_{N-1 \text{ times}}.$$

We may now prove

Theorem 6. *Let* $\bar{\bar{A}} \subseteq R^1$ *be non-empty and compact, and let* $[(\bar{\bar{A}}, U_i); i=1, \ldots, N]$ *be a symmetric game, where* $\forall i$,

$$U_i: \underbrace{\bar{\bar{A}} \times \cdots \times \bar{\bar{A}}}_{N\text{-times}} \to R^1 \quad \textit{is continuous,}$$

except on a subset $A^{**}(i)$ of $A^*(i)$, *where* $A^*(i)$ *is defined by* (2). *Suppose* $\sum_{i=1}^N U_i(\boldsymbol{a})$ *is upper semi-continuous, and* $\forall i$ $U_i(a_i, \boldsymbol{a}_{-i})$ *is bounded and satisfies Property* (α). *Then there exists a symmetric mixed-strategy equilibrium* $(\mu^*, \ldots, \mu^*)$ *with the property that* $\forall i$ *and* $\forall \bar{a}_i \in A_i^{**}(i)$, $\mu^*(\{\bar{a}_i\}) = 0$.

Proof. Let μ^* be the limit of some subsequence of a sequence of probability measures, $\{\mu^n\}$, where

$$\underbrace{(\mu^n, \ldots, \mu^n)}_{N\text{-times}}$$

is a symmetric mixed-strategy equilibrium of the finite game $[(\bar{\bar{A}}^n, U_i); i=1, \ldots, N]$, where $\bar{\bar{A}}^n (n>0)$ is a finite subset of $\bar{\bar{A}}$ with the property $\sup_{a \in \bar{\bar{A}}} \inf_{a^n \in \bar{\bar{A}}^n} |a - a^n| < 1/n$.

By the proof of Lemma 6, we know that $(\mu^*, \ldots, \mu^*)$ is a symmetric mixed-strategy equilibrium of the game $[(\bar{A}, U_i); i=1, \ldots, N]$. Suppose $\exists \bar{a}_i \in A_i^{**}(i)$ such that $\mu^*(\{\bar{a}_i\}) > 0$. Because $U_i(\boldsymbol{a})$ satisfies (α) and $\mu^*(\{\bar{a}_i\}) > 0$, there exists $\lambda \in [0, 1]$ such that

$$\lambda \liminf_{a_i \to \bar{a}_i^-} \int U_i(a_i, \boldsymbol{a}_{-i}) d\mu^*_{-i}(\boldsymbol{a}_{-i}) + (1-\lambda) \liminf_{a_i \to \bar{a}_i^+} \int U_i(a_i, \boldsymbol{a}_{-i}) d\mu^*_{-i}(\boldsymbol{a}_{-i})$$

$$> \int U_i(\bar{a}_i, \boldsymbol{a}_{-i}) d\mu^*_{-i}(\boldsymbol{a}_{-i}).$$

Without loss of generality we may therefore suppose that

$$\liminf_{a_i \to \bar{a}_i^-} \int U_i(a_i, \boldsymbol{a}_{-i}) d\mu^*_{-i}(\boldsymbol{a}_{-i}) > \int U_i(\bar{a}_i, \boldsymbol{a}_{-i}) d\mu^*_{-i}(\boldsymbol{a}_{-i}).$$

But this contradicts the conclusion that μ^* is an equilibrium strategy for player i. ||

Example 2. It will be useful to exemplify Theorem 6 by reviewing games of "timing", or "duels", which have been much studied by game theorists (see Karlin (1959), Owen (1968) and Jones (1980)).[16] These are symmetric two-person zero-sum games on the unit square. The version called the "silent duel" has player 1's payoff function of the form:

$$U_1(a_1, a_2) = \begin{cases} a_1 - a_2 + a_1 a_2 & \text{if } a_1 < a_2 \\ 0 & \text{if } a_1 = a_2 \\ a_1 - a_2 - a_1 a_2 & \text{if } a_1 > a_2. \end{cases}$$

Note first that $A_i^{**}(i) = (0, 1]$ for $i = 1, 2$. Note also that for each $a_2 > 0$, $U_1(a_1, a_2)$ is *left* lower semi-continuous as a function of a_1 at $a_1 = a_2$. Likewise, for each $a_1 > 0$, $U_2(a_1, a_2)$ is *left* lower semi-continuous as a function of a_2 at $a_1 = a_2$. Thus U_i $(i = 1, 2)$ is weakly lower semi-continuous as a function of a_i. Note in particular that U_i satisfies property (α). We conclude that the game satisfies the hypotheses of Theorem 6. Now Theorem 6 says that the game therefore has a symmetric equilibrium where the mixed strategy is atomless on $(0, 1]$. This is indeed the case; for it is known (see Karlin (1959) and Owen (1968)), that the probability density function $p(a_i) = 0$ if $0 \leqq a_i < \frac{1}{3}$ and $p(a_i) = \frac{1}{4} a_i^3$ if $\frac{1}{3} \leqq a_i \leqq 1$ (for $i = 1, 2$) is an equilibrium of this game.

The version which is called the "noisy duel" has player 1's payoff function of the form:

$$U_1(a_1, a_2) = \begin{cases} 2a_1 - 1 & \text{if } a_1 < a_2 \\ 0 & \text{if } a_1 = a_2 \\ 1 - 2a_2 & \text{if } a_1 > a_2. \end{cases}$$

Note first that $A_i^{**}(i) = [0, 1] - \{\frac{1}{2}\}$ for $i = 1, 2$. Note also that for each $a_2 > (<) \frac{1}{2}$, $U_1(a_1, a_2)$ is *left* (*right*) lower semi-continuous as a function of a_1 at $a_1 = a_2$. Likewise, for each $a_1 > (<) \frac{1}{2}$, $U_2(a_1, a_2)$ is *left* (*right*) lower semi-continuous as a function of a_2 at $a_1 = a_2$. Thus U_i $(i = 1, 2)$ is weakly lower semi-continuous in a_i. Clearly U_i satisfies property (α). We conclude that the game satisfies the hypotheses of Theorem 6. It is easy to check (see Karlin (1959) and Owen (1968)) that the pair of pure strategies $a_1 = a_2 = \frac{1}{2}$ is an equilibrium of this game.

6. THE TIGHTNESS OF THE HYPOTHESES OF THEOREM 5

In this section we present examples to demonstrate that the hypotheses of Theorem 5 are "tight", in the sense that dropping any of them renders the theorem invalid.

Observe first that Example 1 in Section 3 fulfils all the hypotheses of Theorem 5 except for the requirement that U_i is weakly lower semi-continuous in a_i. The example therefore demonstrates that some lower semi-continuity condition is essential to guarantee the existence of an equilibrium. We next present examples in which the two other major assumptions are in turn dropped.

Example 3 (dropping the assumption that $\sum_{i=1}^{N} U_i$ is upper semi-continuous). Let $i=1, 2$, let $A_1=A_2=[0, 1]$, and let

$$U_i(a_1, a_2)=\begin{cases} 0 & \text{if } a_1=a_2=1 \\ a_i & \text{otherwise.} \end{cases}$$

The game thus defined satisfies all the hypotheses of Theorem 5 except upper semi-continuity of the sum $\sum U_i$. The game does not possess a mixed-strategy equilibrium because, for any choice of a mixed strategy on the part of one agent that places positive probability on unity, the other will wish to choose a pure strategy as close to unity as possible, but will avoid unity itself. However, if the mixed strategy places zero probability on unity, the other will wish to play unity. Notice finally that the example violates the assumptions of Theorem 5b.

Example 4 (dropping the assumption that the functions f_{ij}^d in the definition of the set $A^*(i)$ are one-to-one). Let $i=1,2$, let $A_1=A_2=[0, 1]$, and let

$$U_1(a_1, a_2)=\begin{cases} a_1, & \text{for } 0\leqq a_1\leqq\frac{1}{2} \text{ and } 0\leqq a_2\leqq\frac{1}{2} \\ a_1+3a_2-2, & \text{for } 0\leqq a_1\leqq\frac{1}{2} \text{ and } 1\geqq a_2>\frac{1}{2} \\ -2a_1+2, & \text{for } 1\geqq a_1>\frac{1}{2} \text{ and } 0\leqq a_2\leqq\frac{1}{2} \\ -2a_1+3a_2, & \text{for } 1\geqq a_1>\frac{1}{2} \text{ and } 1\geqq a_2>\frac{1}{2} \end{cases}$$

and

$$U_2(a_1, a_2)=\begin{cases} a_2, & \text{for } 0\leqq a_1\leqq\frac{1}{2} \text{ and } 0\leqq a_2\leqq\frac{1}{2} \\ -2a_2+2, & \text{for } 0\leqq a_1\leqq\frac{1}{2} \text{ and } 1\geqq a_2>\frac{1}{2} \\ a_2+3a_1-2, & \text{for } 1\geqq a_1>\frac{1}{2} \text{ and } 0\leqq a_2\leqq\frac{1}{2} \\ -2a_2+3a_1, & \text{for } 1\geqq a_1>\frac{1}{2} \text{ and } 1\geqq a_2>\frac{1}{2} \end{cases}.$$

The game thus defined satisfies all the hypotheses of Theorem 5 except for the requirement that the functions f_{ij}^d in the definition of $A^*(i)$ in (2) be one-to-one. Notice that in this example

$$A^*(i)=\{(a_1, a_2)|a_1=\tfrac{1}{2}, 0\leqq a_2\leqq 1\}\cup\{(a_1, a_2)|0\leqq a_1\leqq 1, a_2=\tfrac{1}{2}\},$$

for $i=1,2$. It is clear that the game does not possess an equilibrium. For any choice of a mixed strategy on the part of one player the other will wish to choose a pure strategy as close to $\frac{1}{2}$ from the right as possible but avoiding $\frac{1}{2}$. Note finally that the example violates the assumptions of Theorem 5a.

APPENDIX

In this Appendix we establish the counterparts of the results of this paper for the case where agents' strategy sets are multi-dimensional. Thus, assume $A_i \subseteq R^m$ $(m \geqq 1)$ for $i = 1, \ldots, N$. A typical element of A_i is a_i, and we write a_{ik} for the k-th component of a_i, $(k = 1, \ldots, m)$. Similarly, A_{ik} is the k-th projection of A_i. Otherwise, the notation is the same as in the text.

We begin by characterizing the discontinuity set, as in Section 4.1 of the text. Let $[(A_i, U_i); i = 1, \ldots, N]$ be a game, where $A_i \subseteq R^m$ is non-empty, convex, and compact for all i. Let $Q \subseteq \{1, \ldots, m\}$. For each pair of agents $i, j \in \{1, \ldots, N\}$, let $D(i)$ be a positive integer. For each integer d, with $1 \leqq d \leqq D(i)$ let $f_{ij}^d : R^1 \to R^1$ be a one-to-one, continuous function with the property that $f_{ij}^d = (f_{ji}^d)^{-1}$. Finally, $\forall i$, define

$$A^*(i) = \{(a_1, \ldots, a_N) \in A \mid \exists j \neq i, \exists k \in Q, \exists d, 1 \leqq d \leqq D(i) \text{ such that } a_{jk} = f_{ij}^d(a_{ik})\}. \tag{A1}$$

As in the text, we suppose that $\forall i$ the discontinuities of $U_i(\boldsymbol{a})$ are confined to a subset $A^{**}(i)$ of $A^*(i)$.

In what follows the multi-dimensional version of a given result in the text is denoted by the corresponding starred number.

Lemma 1*. *Let $A_i \subseteq R^m$ $(m \geqq 1)$ be non-empty, convex, and compact for all i, and let $\mu_i \in D(A_i)$. Let $A^*(i) \subseteq A$ be defined by* (A.1) *and let $A^{**}(i) \subseteq A^*(i)$. If $\forall i$, $\forall k \in Q$ and $\bar{a}_{ik} \in A_{ik}$, $\mu_i(\{a_i \in A_i^{**}(i) \mid a_{ik} = \bar{a}_{ik}\}) = 0$, then $\forall i$, $\boldsymbol{\mu}(A^{**}(i)) = 0$. Furthermore, if $\forall \bar{a}_i \in A_i$, $\forall j \neq i$, $\forall d (1 \leqq d \leqq D(i)$ and $\forall k \in Q$, $\mu_j(\{a_j \in A_j \mid a_{jk} = f_{ij}^d(\bar{a}_{ik})\}) = 0$, then $\boldsymbol{\mu}_{-i}(A_{-i}^*(\bar{a}_i)) = 0$, where $A_{-i}^*(\bar{a}_i) = \{\boldsymbol{a}_{-i} \in A_{-i} \mid (\bar{a}_i, \boldsymbol{a}_{-i}) \in A^*(i)\}$.*

Proof. The proof of the first part is virtually identical to its counterpart in the text. In place of the initial inequality (7) we now have

$$\boldsymbol{\mu}(A^{**}(i)) \leqq \sum_{s \neq i} \sum_d \sum_{k \in Q} [(\mu_1 \times \cdots \times \mu_N)(\{(a_1, \ldots, a_N) \in A \mid a_{sk} = f_{is}^d(a_{ik})\} \cap A^{**}(i))]$$

and the remaining argument is the same as in the text.

To prove the second part we note that

$$\begin{aligned}\boldsymbol{\mu}_{-i}(A_{-i}^*(\bar{a}_i)) &\leqq \sum_{j \neq i} \sum_d \sum_{k \in Q} \boldsymbol{\mu}_{-i}(\{\boldsymbol{a}_{-i} \in A_{-i} \mid a_{jk} = f_{ij}^d(\bar{a}_{ik})\}) \\ &\leqq \sum_{j \neq i} \sum_d \sum_{k \in Q} u_j(\{a_j \in A_j \mid a_{jk} = f_{ij}^d(\bar{a}_{ik})\}) \prod_{s \neq i,j} \mu_s(A_s) = 0. \quad \|.\end{aligned}$$

The proof of Lemma 2 in the text does not assume that A_i is one-dimensional. It therefore applies without change to the multi-dimensional case. We may now state

Theorem 4*. *For $i = 1, \ldots, N$, let $A_i \subseteq R^m (m \geqq 1)$ be non-empty, convex, and compact, and let $U_i : A \to R^1$ be continuous, except on a subset of $A^*(i)$, where $A^*(i)$ is defined by* (A.1). *Suppose, $\forall i$, U_i is bounded. Let A_i^n be a finite subset of A_i with the property that*

$$\sup_{a_i \in A_i} \inf_{a_i^n \in A_i^n} \|a_i - a_i^n\| < 1/n.$$

Let $\boldsymbol{\mu}^n = (\mu_1^n, \ldots, \mu_N^n)$ be an equilibrium vector of mixed strategies for the finite game $[(A_i^n, U_i); i = 1, \ldots, N]$, and let $\boldsymbol{\mu}^n \to \boldsymbol{\mu}^*$ as $n \to \infty$; (see footnote 12 *in the text). Suppose $\forall i, j (i \neq j)$, $\forall d$, $\forall k \in Q$ and $\forall \bar{a}_i \in A_i^{**}(i)$, $\mu_j^*(\{a_j \in A_j \mid a_{jk} = f_{ij}^d(\bar{a}_{ik})\}) = 0$. Then $\boldsymbol{\mu}^* = (\mu_1^*, \ldots, \mu_N^*)$ is an equilibrium for the infinite game $[(A_i, U_i); i = 1, \ldots, N]$.*

Proof. Identical to the proof of Theorem 4 in the text. ||

We turn next to the generalization of Lemma 3.

Lemma 3*. *Let $A_i \subseteq R^m$ $(m \geqq 1)$, $(i = 1, \ldots, N)$, be non-empty, convex, and compact, and let $U_i: A \to R^1$ be continuous, except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by* (A1). *Suppose $\forall i$, U_i is bounded. Let A_i^n be a finite subset of A_i with the property that*

$$\sup_{a_i \in A_i} \inf_{a_i^n \in A_i^n} \|a_i - a_i^n\| < 1/n.$$

Let $(\mu_1^n, \ldots, \mu_N^n)$ be an equilibrium vector of mixed strategies for the finite game $[(A_i^n, U_i); i = 1, \ldots, N]$, and let $(\mu_1^*, \ldots, \mu_N^*) = \lim_{n\to\infty} (\mu_1^n, \ldots, \mu_N^n)$; (see footnote 12). Then $\forall i$, a set of pure strategies $\bar{a}_i \in A_i$ that differ from one another in every component $\bar{a}_{ik}$[17] $(k = 1, \ldots, m)$ and that satisfy

$$\int_{A_{-i}} U_i(\bar{a}_i, \boldsymbol{a}_{-i}) d\boldsymbol{\mu}_{-i}^* > \lim_{n\to\infty} \int_A U_i(\boldsymbol{a})\, d\boldsymbol{\mu}^n \tag{A2}$$

contains at most countably many members.

Proof. Suppose $\exists i$ for which there are uncountably many pure strategies that differ from one another in every component and that satisfy (A2). By Lemma 2 we conclude that $\boldsymbol{\mu}_{-i}^*(\bar{A}_{-i}^{**}(\bar{a}_i)) > 0$ for each such $\bar{a}_i$, where $\bar{A}_{-i}^{**}(\bar{a}_i)$ is the closure of the set $A_{-i}^{**}(\bar{a}_i) = \{\boldsymbol{a}_{-i} \in A_{-i} | (\bar{a}_i, \boldsymbol{a}_{-i}) \in A^{**}(i)\}$. But $\bar{A}_{-i}^{**}(\bar{a}_i) \subseteq A_{-i}^*(\bar{a}_i)$. Therefore, from Lemma 1* we can conclude that for each such $\bar{a}_i$, $\exists j \neq i$, $\exists d$ $(1 \leqq d \leqq D(i))$ and $\exists k \in Q$ such that

$$\mu_j^*(\{a_j \in A_j | a_{jk} = f_{ij}^d(\bar{a}_{ik})\}) > 0. \tag{A3}$$

Thus, there exist j and d and k such that for uncountably many a_i $(i \neq j)$, (A3) holds. But the functions f_{ij}^d are by hypothesis one-to-one. Therefore, the sets $\{a_j \in A_j | a_{jk} = f_{ij}^d(\bar{a}_{ik})\}$ are disjoint. We thus conclude that there are uncountably many disjoint subsets of A_j of positive measure, a contradiction. ||

In what follows we shall consider games which satisfy the hypotheses of Lemma 3* and for which $\sum_{i=1}^N U_i(\boldsymbol{a})$ is upper semi-continuous. We now generalize the (weakly) lower semi-continuous property (see Definition 6). Let B^m be the surface of the unit sphere in R^m with the origin as its centre. Let $e \in B^m$, and let θ be a positive number. Then we say that $U_i(a_i, \boldsymbol{a}_{-i})$ is weakly lower semi-continuous in a_i if for all $\bar{a}_i \in A_i^{**}(i)$ there exists an absolutely continuous measure ν on B^m such that for all $\boldsymbol{a}_{-i} \in A_{-i}^{**}(\bar{a}_i)$,

$$\int_{B^m} [\liminf_{\theta\to 0} U_i(\bar{a}_i + \theta e, \boldsymbol{a}_{-i}) d\nu(e)] \geqq U_i(\bar{a}_i, \boldsymbol{a}_{-i}).$$

Lemma 4*. *$\forall i$, let $A_i \subseteq R^m$ $(m \geqq 1)$ be non-empty, convex and compact, and let $U_i: A \to R^1$ $(i = 1, \ldots, N)$ be continuous, except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by* (A1). Suppose *$\sum_{i=1} U_i(\boldsymbol{a})$ is upper semi-continuous, $U_i(a_i, \boldsymbol{a}_{-i})$ is bounded and is weakly lower semi-continuous in a_i. Let A_i^n be a finite subset of A_i with the property that*

$$\sup_{a_i \in A_i} \inf_{a_i^n \in A_i^n} \|a_i - a_i^n\| < 1/n.$$

Let $(\mu_1^n, \ldots, \mu_N^n)$ be an equilibrium vector of mixed strategies for the finite game $[(A_i^n, U_i); i = 1, \ldots, N]$, and let $(\mu_1^*, \ldots, \mu_N^*) = \lim_{n\to\infty} (\mu_1^n, \ldots, \mu_N^n)$. Then $\forall i$

$$\lim_{n\to\infty} \int_A U_i(\boldsymbol{a}) d\boldsymbol{\mu}^n = \int_A U_i(\boldsymbol{a}) d\boldsymbol{\mu}^*. \tag{A4}$$

Proof. Suppose $\exists i$ for which (A4) does not hold. Then by an argument identical to the one in Lemma 4 we conclude that $\exists j$, $\exists \bar{a}_j \in A_j$ and $\exists \varepsilon > 0$ such that

$$\int_{A_{-j}} U_j(\bar{a}_j, \boldsymbol{a}_{-j})\, d\boldsymbol{\mu}^*_{-j} > \lim_{n\to\infty} \int U_j(\boldsymbol{a})\, d\boldsymbol{\mu}^n + \varepsilon. \tag{A5}$$

But U_j is weakly lower semi-continuous in a_j. Therefore from (A5) we can conclude that

$$\int_{B^m} \left[\liminf_{\theta\to 0} \int_{A_{-j}} U_j(\bar{a}_j + \theta e, \boldsymbol{a}_{-j})\, d\boldsymbol{\mu}^*_{-j}\right] d\nu(e) > \lim_{n\to\infty} \int U_j(\boldsymbol{a})\, d\boldsymbol{\mu}^n + \varepsilon. \tag{A6}$$

But ν is an absolutely continuous distribution on B^m. Therefore, (A6) implies that there exist uncountably many pure strategies, $\tilde{a}_j$, that differ from one another in every component k and that satisfy

$$\int_{A_{-j}} U_j(\tilde{a}_j, \boldsymbol{a}_{-j})\, d\boldsymbol{\mu}^*_{-j} > \lim_{n\to\infty} \int U_j(\boldsymbol{a})\, d\boldsymbol{\mu}^n.$$

This contradicts Lemma 3*. ||

Theorem 5*. *For all i, let $A_i \subseteq R^m$ $(m \geqq 1)$ be non-empty, convex and compact, and let $U_i: A \to R^1$ be continuous except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by* (A1). *Suppose $\sum U_i(\boldsymbol{a})$ is upper semi-continuous, for all i $U_i(a_i, \boldsymbol{a}_{-i})$ is bounded and is weakly lower semi-continuous in a_i. Then the game $[(A_i, U_i); i = 1, \dots, N]$ possesses a mixed strategy equilibrium.*

Proof. Simple adaptation of the proof of Theorem 5, but using Lemmas 3* and 4* instead of 3 and 4. ||

In the text, symmetric games have been defined for the case where agents' strategy sets are multi-dimensional. Indeed, the existence of symmetric mixed-strategy equilibria for symmetric games which satisfy the hypotheses of Theorem 5* follows directly from Lemma 5 and Theorem 5*. We proceed, therefore, to a generalization of Theorem 6. We first state the multi-dimensional version of condition (α). We call this (α^*).

Property (α^*). $\forall \bar{a}_i \in A_i^{**}(i)$, $\exists$ a non-atomic measure ν on B^m such that for all $\boldsymbol{a}_{-i} \in A_{-i}^{**}(\bar{a}_i)$

$$\int_{B^m} [\lim_{\theta\to 0}\inf U_i(\bar{a}_i + \theta e, \boldsymbol{a}_{-i})\, d\nu(e)] \geqq U_i(\bar{a}_i, \boldsymbol{a}_{-i}),$$

where the inequality is strict if

$$\boldsymbol{a}_{-i} = \underbrace{(\bar{a}_i, \dots, \bar{a}_i)}_{(N-1)\text{ times}}.$$

We may now state

Theorem 6*. *Let $\bar{\bar{A}} \subseteq R^m$ $(m \geqq 1)$ be non-empty, convex and compact, and let $[(\bar{\bar{A}}, U_i); i = 1, \dots, N]$ be a symmetric game, where $\forall i$,*

$$U_i: \underbrace{\bar{\bar{A}} \times \cdots \times \bar{\bar{A}}}_{N\text{ times}} \to R^1$$

is continuous, except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by (A1). *Suppose $\sum_{i=1}^N U_i(\boldsymbol{a})$ is upper semi-continuous, and for all i, U_i is bounded and satisfies Property* (α^*).

Then there exists a symmetric mixed-strategy equilibrium $(\mu^*, \ldots, \mu^*)$ with the property that $\forall i$ and $\forall \bar{a}_i \in A_i^{**}(i)$, $\mu^*(\{\bar{a}_i\}) = 0$.

Proof. Virtually identical to the proof of Theorem 6. ||

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NOTES

1. The non-existence problem may therefore appear to be more complex than in the examples exhibited by McManus (1964) and Roberts and Sonnenschein (1977), where payoff functions fail to be quasi-concave but are continuous. See, however, Dierker and Grodal (1982), who show that there are examples in the Roberts-Sonnenschein framework where discontinuities are so severe that even a mixed strategy equilibrium fails to exist.

2. The games of timing that have been most frequently discussed are "duels" (see Karlin (1959), Owen (1968) and Jones (1980)). The payoff functions in these games are neither upper nor lower semi-continuous, and so Glicksberg's (1950) theorem does not apply to them. (See example 2 in Section 5 below.)

3. Rosenthal (1980) also demonstrates, by construction, the existence of a symmetric mixed strategy equilibrium in a model of oligopolistic international trade.

4. R^m is m-dimensional Euclidean space ($m \geqq 1$).

5. $\psi_i: A_{-i} \to A_i$ is *upper hemi-continuous* if $\{\boldsymbol{a}_{-i}^n\} \subseteq A_{-i}$ and $\boldsymbol{a}_{-i}^n \to \bar{\boldsymbol{a}}_{-i}$; $a_i^n \in \psi_i(\boldsymbol{a}_{-i}^n)$ and $a_i^n \to \bar{a}_i$ imply $\bar{a}_i \in \psi_i(\bar{\boldsymbol{a}}_{-i})$. An upper hemi-continuous and compact-valued correspondence is sometimes called a *closed* correspondence.

6. It is simple to show that graph continuity and u.s.c. of U_i imply that V_i is l.s.c.

7. In what follows we shall always endow $D(A_i)$ with the topology of weak convergence; i.e. a sequence of measures $\{\mu_i^n\} \subseteq D(A_i)$ converges to $\mu_i^* \in D(A_i)$ if $\int_{A_i} f(a_i)d\mu_i^n \to \int_{A_i} f(a_i)d\mu_i^*$ for all real-valued continuous functions f defined on A_i.

8. $(\mu_1^*, \ldots, \mu_i^*, \ldots, \mu_N^*)$ is an ε-equilibrium if for all i, μ_i^* results in an expected payoff to agent i not more than ε less than the supremum when j ($j \neq i$) plays μ_j^*.

9. In the Appendix we consider games in which (pure) strategy sets are multi-dimensional.

10. To be precise, $\boldsymbol{\mu}^*$ is the limit of some subsequence $\{\boldsymbol{\mu}^{n_k}\}$.

11. Note that if U_i were continuous on A_i, (5) would hold with equality by the definition of the topology of weak convergence of probability measures.

12. We know that $\{\boldsymbol{\mu}^n\}$ possesses a sub-sequence which converges. Thus we are supposing that $\boldsymbol{\mu}^*$ is the limit of some sub-sequence of $\{\boldsymbol{\mu}^n\}$.

13. From the boundedness of U_i we may conclude that a subsequence of $\{\int U_i(\boldsymbol{a})d\boldsymbol{\mu}^n\}$ converges. Thus, without loss of generality we suppose that $\lim_{n\to\infty} \int U_i(\boldsymbol{a})d\boldsymbol{\mu}^n$ exists.

14. "$a_i \xrightarrow{-} \bar{a}_i$" (resp. "$a_i \xrightarrow{+} \bar{a}_i$") denotes that a_i approaches $\bar{a}_i$ from the left (resp. right). Notice that if $U_i(a_i, \boldsymbol{a}_{-i})$ is lower semi-continuous in a_i, it is weakly lower semi-continuous in a_i.

15. This result is well known among game theorists. We are including a proof here because we have been unable to find a reference. We are indebted to Hervé Moulin for suggesting the use of the Fan lemma here.

16. We are grateful to J.-F. Mertens for drawing our attention to these games.

17. By differing in every component, we mean that if $\bar{a}_i$ and $\bar{a}_i'$ are two such pure strategies, then $\bar{a}_{ik} \neq \bar{a}_{ik}'$ for all k.

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