Election Methods and Strategic Voting

E. Maskin and R. Robinette

April 2024

Voting Theory

- interesting intellectual exercise
- provides foundation for *voting reform*
  - in U.S., recent progress in replacing *plurality rule* (each voter for one candidate, candidate with most votes wins) with better rules such as *ranked-choice voting*
  - theoretical and empirical work has guided past reform and likely to guide future reform
- in addition to its academic contributions, today's paper is very much geared to reform

good place to start: Arrow and Gibbard-Satterthwaite impossibility theorems

- Arrow Theorem: no voting rule satisfies
  - Unrestricted domain (U) – election always produces winner
  - Pareto principle (P) – if all voters prefer *x* to *y*, *y* won't be elected
  - Anonymity (A) – all voters treated equally
  - Neutrality (N) – all candidates treated equally
  - Independence of Irrelevant Alternatives (IIA) – whether *x* or *y* wins shouldn't depend on what voters think of *z*
- Maskin (2024) argues that IIA as formulated by Arrow is unjustifiably stringent
- when relaxed appropriately (and P strengthened slightly)
  - impossibility vanishes
  - unique characterization of *Borda Count*
    - candidate gets *m* points every time ranked first, *m* – 1 points when ranked second, etc.
    - candidate with most points wins

Today's focus on:

- Gibbard-Satterthwaite Theorem: no voting rule satisfies
  - Unrestricted domain (U)
  - Anonymity (A)
  - Neutrality (N)
  - Strategy proofness (SP)
    - voters don't have incentive to rank candidates untruthfully
- will question assumption U

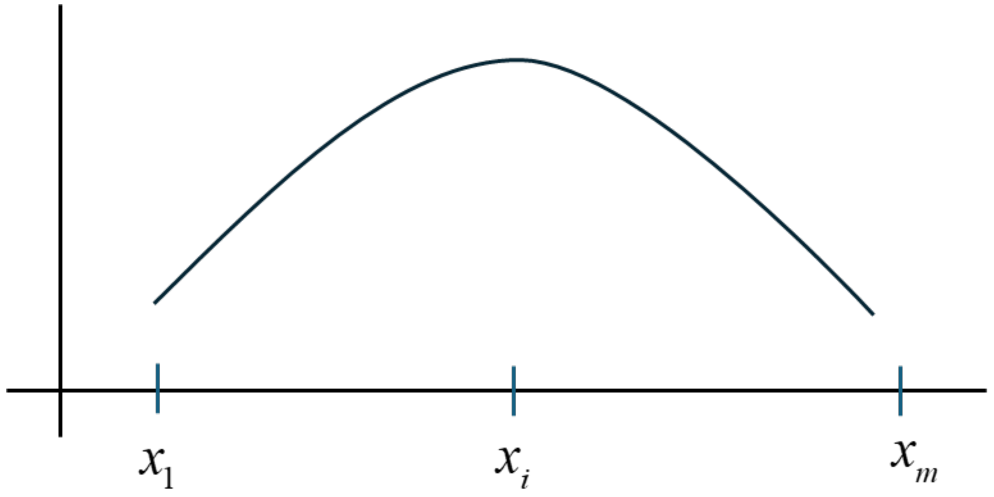
- Two reasons why an election system resistant to strategic voting is desirable
  - it ensures that winner is appropriate given voters’ actual preferences
  - it makes life easier for voters

- Gibbard-Satterthwaite Theorem says that can’t achieve SP always
- However, it’s been known at least since Dummett-Farquharson (1961) that for “natural” restrictions on preferences, strategy-proofness can be achieved
  - in particular, if preferences are single-peaked then majority rule is strategy-proof

- *single-peaked preferences*:
  - there exists an ordering of the candidates $x_1,...,x_m$ such that if

  (i) if $x_i \succ x_j$, for $i < j$, then $x_j \succ x_k$ for $j < k$

  (ii) if $x_i \succ x_j$ for $i > j$, then $x_j \succ x_k$ for $j > k$
  - so, if we graph utility, it has single peak



- *majority rule* (Condorcet voting)
  - given voters' preferences, elect the candidate x such that, for all $y \neq x$ number of voters preferring x to y > number of voters preferring y to x
    - x is *Condorcet winner*
- Condorcet showed by example that Condorcet winner may not exist

33%	**34%**	**33%**
x	y	z
y	z	x
z	x	y

  - x beats y
  - z beats x
  - y beats x

- But Black (1944) showed that, if preferences single-peaked, Condorcet winner exists.

- consider peak for each voter

  let x_{i_*} be *median candidate*

  - then 50% of voters have their peak to right of x_{i_*}
  - all those voters prefer x_{i_*} to any candidate to left of x_{i_*} (by single-peakedness)
  - so, majority prefer x_{i_*} to any candidate to left of x_{i_*}
  - similarly, majority prefer x_{i_*} to any candidate to right of x_{i_*}
  - so x_{i_*} is Condorcet winner

- would any voter gain from voting strategically?
  - i.e., submitting false preference?
- suppose voters just submit their *peaks* so Condorcet winner is median peak x_{i_*}
  - important restriction (as we'll see)
- suppose that a voter's peak is to *left* of x_{i_*}
  - if reports peak to left of x_{i_*}, then x_{i_*} remains median – so no effect on outcome
  - if reports peak to right of x_{i_*}, then median moves to *right* of x_{i_*}
    - but given single-peaked preferences, this is *worse* outcome for voter than x_{i_*}
- so voting truthfully is dominant strategy

This result has some useful applications

- consider n citizens voting over public good level x
  - x costs $c(x)$, where c increasing and convex function
  - each citizen pays $c(x)/n$
  - citizen a prefers x_a that solves $x_a = \underset{x}{\operatorname{argmax}}\, u_a\left(x, -c(x)/n\right)$
- then citizen a's preferences single-peaked over $\{x_1, ..., x_n\}$, where $x_1 < x_2 < ... < x_n$
- suppose each voter submits peak (favorite level)
- winner is median level (Condorcet winner)
- voters have incentive to vote truthfully
- this method proposed by Bowen (1943)

- can we apply idea to political elections?
- there is a lot of evidence (which we will come to) that voters preferences *are* single-peaked (with two qualifications that we’ll discuss)
- so if candidates are Bernie Sanders, Michael Bloomberg, Donald Trump, we’d expect
  - Sanders voters prefer Bloomberg to Trump
  - Trump voters prefer Bloomberg to Sanders

- in public good example, we can arrange levels numerically, $x_1 < x_2 < x_3$, have voters vote for their favorite level and elect the median level
- in political example, suppose we designate
  - Sanders as left-wing candidate
  - Bloomberg as centrist
  - Trump as right-wing

  have each voter vote for favorite candidate and elect median (on left/center/right scale)
- but conducting an election like this in reality *impossible*
  - designating Bloomberg as the centrist in advance gives him an enormous advantage
  - election method is highly *nonneutral*
    - neutrality requires that if we permute the candidates so that $x \to y \to z \to x$, then if x won before, now y wins

- suppose voters report *rankings* of candidates
- Condorcet winner on basis of rankings elected

- suppose

46%	**20%**	**5%**	**29%**
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

- if voters report rankings truthfully, then

71%	**54%**
Bloomberg	Bloomberg
Trump	Sanders

  – Bloomberg easily wins

46%	20%	5%	29%
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

- However, suppose Sanders voters report

  Sanders
  Trump
  Bloomberg

- Then

66%	54%	75%
Sanders	Bloomberg	Trump
Trump	Sanders	Bloomberg

  – so no Condorcet winner
  – have to use “tiebreaking” rule
  – large literature on tiebreaking rules

46%	**20%**	**5%**	**29%**
Sanders	Bloomberg	Bloomberg	Trump
Trump	Sanders	Trump	Bloomberg
Bloomberg	Trump	Sanders	Sanders

- suppose we use Borda count to break tie (per Black 1958)
  - candidate gets 2 points every time ranked first
  - candidate gets 1 point every time ranked second
  - candidate gets 0 points every time ranked third
  - winner is candidate with most points
- Sanders has 112 points, Trump 109 points, Bloomberg 79 points
  - Sanders wins
  - strategic voting by Sanders votes *pays off*
- similarly, can show *all other* standard tie breakers can be strategically manipulated

46%	**20%**	**5%**	**29%**
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

- so is there way to elect Condorcet winner in a strategy-resistant way?
- notice there is more "diversity" among Bloomberg supporters than among Sanders or Trump supporters
  - – let's try to exploit this endogenous feature of preferences

46%	20%	5%	29%
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

- let's begin with ranked-choice voting (RCV)
  - voters rank candidates
  - the candidate with majority of first-place votes is elected
  - if no such candidate, drop candidate ranked first least often, elevate his supporters' second choices into first, and iterate process until someone has majority
  - now used in many American cities and states of Maine and Alaska
- so in example, Bloomberg is first eliminated and then Sanders wins with 66% of vote
- Now, RCV is easily manipulated by strategic voting
  - in above example, if Trump voters submit ranking

    Bloomberg
    Trump
    Sanders

    then Bloomberg (rather than Sanders wins)
    - Trump voters prefer Bloomberg to Sanders

46%	**20%**	**5%**	**29%**
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

- But suppose we modify RCV to take into account a candidate's *diversity score*: the number of *different* rankings in which voters rank the candidate first
  - thus, in example, Bloomberg has a diversity score of 2
  - Trump and Sanders have diversity scores of 1
  - diversity score is a measure of how *varied* a candidate's support is

46%	20%	5%	29%
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

- Election rules
  - – each voter ranks the candidates
  - – if some candidate has majority of first-place vote, she’s elected
  - – if no such candidate, candidate with lowest diversity score is dropped
  - – if tie for lowest diversity, one with fewest first-place votes dropped
  - – process continues until some candidate has majority of first-place votes
- In example,
  - – Trump and Sanders tie for lowest diversity score
  - – Trump has fewer first-place votes, so dropped
  - – then Bloomberg has majority of first-place votes, so elected

- Assumptions
  - (1) each voter's truthful ranking is single-peaked on left-right continuum.
  - (2) each possible single-peaked ranking is a truthful ranking for some voters (fullness)
  - (3) a coalition of voters who vote strategically will coordinate on single (strategic) ranking, i.e., all coalition members submit same ranking
- *Theorem*: Consider the above election under assumptions (1) – (3), no coalition of voter gains from voting strategically if it assumes truthful preferences constitute full single-peaked profile
  - not quite strategy-proofness
    - truthful voting may not be dominant strategy against non-single-peaked rankings or non-full profile
    - coalition submits only one strategic ranking

Proof (for case of 3 candidates Sanders, Bloomberg, Trump):

Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

- From assumptions (1) and (2), profile must look like above
- *Case I*: Sanders is truthful Condorcet winner
  - since majority must prefer Sanders to Bloomberg, majority have ranking

    Sanders
    Bloomberg
    Trump

  - so no manipulation worthwhile

Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

Case II: Bloomberg truthful Condorcet winner

- with truthful voting, Bloomberg has diversity score 2, and Sanders and Trump have diversity scores 1
- could Sanders-supporters get Sanders elected by voting strategically?
  - could increase Sanders's diversity score to 2 by submitting

    Sanders

    Trump

    Bloomberg

    - then Trump dropped (if Bloomberg doesn't win outright), but Bloomberg is then ranked first by majority
  - could increase Trump's diversity score by submitting

    Trump

    Sanders

    Bloomberg

    - but then Sanders dropped, which is clearly not profitable

- Given difficulties of coordinating strategic behavior assumption (3) seems plausible
  - also, would be *dangerous* for Sanders voters to submit Trump
    Sanders
    Bloomberg

- also (3) necessary for Theorem:

46%	**20%**	**5%**	**29%**
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

  - suppose Sanders-voters submit both

Sanders	Trump
Trump	Sanders
Bloomberg	Bloomberg

  - then Sanders, Bloomberg, and Trump all have diversity score 2
  - so Bloomberg dropped
  - then Sanders has majority, and so is elected
    - this is successful manipulation

Voting system is *unique* in three senses

- if alternative voting rule elects a Condorcet winner for any single-peaked profile and is resistant to strategic voting for full single-peaked profiles, then alternative rule must produce the same winner as our voting rule for all deviations from truthful voting
- if a voting rule satisfying A and N is strategy-resistant for full single-peaked profiles, it must elect a Condorcet winner for truthful voting
- there is no voting rule satisfying P, A, and N that is resistant to strategic voting on a superset of the single-peaked preferences

46%	20%	5%	29%
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

proof that deviation must lead to same outcome as our voting rule:

- consider profile on top of slide
  - winner is Bloomberg
- suppose some Sanders-supporters submit

  Sanders

  Trump

  Bloomberg

  - under voting rule, Bloomberg still elected
  - if deviation causes Sanders to be elected, then election rule not strategy resistant
  - if deviation causes Trump to be elected then some Trump-supporters will submit this ranking instead, violating strategy resistance

46%	20%	5%	29%
Sanders	Bloomberg	Bloomberg	Trump
Bloomberg	Sanders	Trump	Bloomberg
Trump	Trump	Sanders	Sanders

Proof that voting rule satisfying A and N and strategy-resistant on full profiles must be Condorcet

- suppose Sanders (not Bloomberg) is winner for profile above
- consider

46%	25%	29%
Sanders	Bloomberg	Trump
Bloomberg	Sanders	Bloomberg
Trump	Trump	Sanders

  – Sanders wins (otherwise Bloomberg / Trump / Sanders voters will manipulate)

$(*)$

46%	54%
Sanders	Bloomberg
Bloomberg	Sanders
Trump	Trump

  – Sanders wins

From N

$(**)$

46%	54%
Bloomberg	Sanders
Sanders	Bloomberg
Trump	Trump

  – Bloomberg wins

- but Sanders voters can convert $(**)$ to $(*)$

- Are assumptions about single-peakedness realistic for political elections?
  - answer: yes with two caveats

Here's a typical example: 2022 special election for U.S. House of Representatives in Alaska

- 3 candidates: Sarah Palin (Trumpist), Nick Begich (moderate Republican), Mary Peltola (Democrat)
- here is distribution of rankings (we know them because Alaska uses RCV):

18%	**11%**	**2%**	**14%**	**6%**	**8.5%**	**25%**	**13%**	**2.5%**
Palin	Palin	Palin	Begich	Begich	Begich	Peltola	Peltola	Peltola
Begich		Peltola	Palin		Peltola	Begich		Palin

- Two deviations from our assumptions
  - some voters didn't rank candidates -- just voted for favorite candidates (call these rankings *weakly* single-peaked)
  - *all* rankings (not just single-peaked) rankings are represented
    - however, fractions of non-single-peaked preferences tiny (2% and 2.5%)

18%	11%	2%	14%	6%	8.5%	25%	13%	2.5%
Palin	Palin	Palin	Begich	Begich	Begich	Peltola	Peltola	Peltola
Begich		Peltola	Palin		Peltola	Begich		Palin

- weak single-peakedness poses potential problem for existence of Condorcet winner

$1/3$	3ε	$1/3-5\varepsilon$	$1/3+2\varepsilon$
Sanders	Bloomberg	Bloomberg	Trump
	Sanders		Bloomberg

– get

$1/3+2\varepsilon$	$1/3-2\varepsilon$
Trump	Bloomberg
Bloomberg	Trump

$2/3$	$1/3$
Bloomberg	Sanders
Sanders	Bloomberg

$1/3+3\varepsilon$	$1/3+2\varepsilon$
Sanders	Trump
Trump	Sanders

– Condorcet cycle

- but this can’t happen if proportion of strict single-peakedness exceeds weak single-peakedness

Theorem extends to weak single-peakedness and non-single-peakedness as long as not too much of either

- weak single-peakedness exceeded by strict single-peakedness
- non-single-peakedness below threshold