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Withholding and damage in Bayesian trade mechanisms ☆

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A B S T R A C T

We study the optimality of allowing the designer to withhold or damage resources in Bayesian incentive compatible mechanisms for bilateral trade with independent private values. The following results hold when the buyer and the seller have discrete value distributions. Burning money or withholding the good from both traders never enhances welfare. Similarly, damaging the good for the buyer cannot increase welfare. By contrast, damaging the good for the seller may improve welfare. However, such welfare improvements are possible only if the damage hurts some lower valuation type of seller more severely than the highest valuation type. Results extend to the case of continuous value distributions under certain hypotheses regarding virtual values. Methods also apply to optimal Bayesian implementation for allocation problems. In the absence of property rights, damaging goods for any agent has negative welfare consequences.

1. Introduction

A major theme of the literature on mechanism design with asymmetric information is that to achieve an optimal allocation of resources, it may be necessary to *deliberately* introduce distortions that would be inefficient in a complete information setting. For example, consider a seller who has an indivisible good to which she attaches no value facing a buyer whose valuation for the good is equally likely to be 3 or 7. In the mechanism that maximizes the seller's expected revenue, the seller sets a price of 7, and the buyer acquires the good when his valuation is 7, but not when it is 3. This outcome is inefficient because either buyer type derives greater value from the good than the seller. The inefficiency arises because by not trading with the low-value buyer, the seller is able to demand a high price from the high-value buyer. If the valuation of the high-type buyer were reduced from 7 to 5, then the revenue maximizing mechanism prescribes that the seller trade with both buyer types at price 3. The reduction in the value of the high-type buyer not only increases the probability of an efficient allocation under the revenue maximizing mechanism from 1/2 to 1, but also increases total expected welfare (the sum of expected buyer and seller payoffs) from 7/2 to 4.

Going a step further, Green and Laffont (1979), Laffont and Maskin (1979) and Myerson and Satterthwaite (1983) exhibit models in which, because of asymmetric information, there is *no* mechanism that implements efficient allocations and satisfies natural participation and budget balance constraints. For an illustration, consider the following instance of the Myerson-Satterthwaite model of bilateral trade with incomplete information. A seller owns an indivisible good, which she

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values at either $s_1 \in (0, 1)$ or 3 with equal probability. A buyer is interested in acquiring the good, and values it at either 1 or 4, also with equal probability. The valuations of the seller and the buyer are independently distributed. Efficiency dictates that the good be sold with probability 1 for the pairs of buyer and seller types $(1, s_1), (4, s_1), (4, 3)$ and with probability 0 for the pair $(1, 3)$. However, no Bayesian incentive compatible mechanism that is interim idividually rational and ex post budget balanced delivers these outcomes. The welfare maximizing mechanism in this class entails that for the pair of types $(1, s_1)$, the good is traded with probability $2/(2 + s_1)$. In this example, reducing the value s_1 of the low-type seller increases the probability that the good is allocated efficiently in the optimal mechanism, and also enhances total welfare if $s_1 \leq \sqrt{6} - 2$.

The examples above suggest that reducing valuations and, more generally, creating inefficiencies at the ex ante stage of a mechanism may improve ex post allocative efficiency and social welfare. In this paper, we explore the possibility of such welfare improvements in the Myerson-Satterthwaite bilateral trade setting when the mechanism designer has access to two types of inefficient actions: withholding and damage.

Under *withholding*, we imagine that when the good is handed over by the seller, it does not necessarily reach the buyer: the mechanism designer can withhold or destroy the good with some probability. Similarly, the mechanism designer has the option to withhold part of the monetary transfer from the buyer to the seller (i.e., "burn money" or run an ex post budget surplus). We assume that the mechanism designer can condition both types of withholding on traders' reports.

When the good is deliberately *damaged*, it is reduced in value for one or more buyer or seller types. This may entail physical damage that is intentional or exogenous. Regulations that restrict the use of the good or prohibit bundling may also reduce a trader's valuation and have idiosyncratic effects on types (whereas property taxes reduce valuations uniformly across types). For a concrete example, suppose that the good being sold is a private airplane that has a navigational device attuned to the seller's geographic region. The buyer lives in a different region, and removing the device will not affect him at all. However, removing it will impact a seller with low valuation more because higher-valuation seller types are intensely invested in flying and have other equipment that could substitute for the missing device. Similarly, a broker handling an estate or an art collection can decide which items should be included for sale as a package, and dictate what the seller should do with the excluded items prior to the sale.

We begin our analysis with the case in which the two traders have discrete value distributions. This setting helps us develop intuition and yields the strongest versions of the results. The general principle behind creating inefficiencies is that they may enhance a mechanism's ability to discriminate between different agent types. Since our model (as much of the mechanism design literature) assumes quasilinear preferences, there is no difference between types in their marginal utility for money. Hence, there should be no welfare gain from withholding money, and this intuition is borne out by our Proposition 1.

More interestingly, Proposition 2 establishes that withholding the good never improves welfare either. To understand the issues involved in establishing this seemingly intuitive result, note that if a mechanism withholds the good for some pair of buyer and seller types, the natural path to welfare improvement requires increasing the probability that the good is allocated to either of these types. However, this perturbation may result in a violation of the monotonicity of traders' probability of being allocated the good with respect to their types, which is a necessary condition for implementation. In this situation, we show how the allocation can be further perturbed via a sequence of changes in trading probabilities that has a neutral effect on the overall probability that each buyer and seller type affected by the changes is allocated the good until we eventually reach a type for which the monotonicity condition is not binding. With suitable adjustments in monetary transfers, the perturbed allocation is implementable and improves welfare over the original one.

Proposition 3 shows that damaging the good for the buyer is never socially optimal. The intuition is that reducing buyer valuations entails less scope for allocating the good to the buyer and thus decreases potential gains from trade.

By contrast, damaging the good for the seller may enhance social welfare, as our second opening example demonstrates. This is true even if the seller needs to consent to the damage, which imposes the constraint that the utility she obtains from participating in the mechanism should be greater than or equal to her value for the *undamaged* good. Proposition 4 synthesizes the potential welfare benefits of damage for the seller. The idea is that a seller type whose valuation has been reduced finds it less attractive to pretend to be a higher-type seller. The slack created in the underlying incentive constraint can be leveraged to increase the probability of trade for seller types with reduced valuations. Proposition 4 suggests that it may be valuable to employ a mechanism designer who is more "active" than usually considered.

However, Proposition 4 relies on the mechanism designer damaging the good to a greater extent for seller types that value it less. In some situations, the mechanism designer may be constrained to damage the good in a fashion that affects higher-valuation seller types equally or more. Then, Proposition 5 reverses the conclusion of Proposition 4 under the maintained assumption that the seller should weakly prefer trading the damaged good in the mechanism to walking away with the undamaged good: damage that affects the seller type with the highest valuation most severely cannot improve welfare. The proof of Proposition 5 shows that even though reduced valuations for lower seller types decrease the information rents commanded by the seller as highlighted in Proposition 4, the transfers the designer saves on lower types by damaging the good are not sufficient to compensate the highest valuation seller type for participating in the mechanism and running the risk of being allocated the damaged good.

We develop parallel results for the original Myerson-Satterthwaite setting in which the buyer and the seller have continuous value distributions. Propositions 6, 7, 8 and 9 extend Propositions 1, 2, 3 and 5, respectively, under mild conditions regarding virtual valuations. We also briefly consider related questions for social choice rules and allocation problems without initial ownership as modeled by Wolitzky (2016).

There are several contributions related to our paper. Cramton et al. (1987) show that the tension between incomplete information and allocative efficiency highlighted by the impossibility theorem of Myerson and Satterthwaite is remediated by dispersing property rights among agents. Matsuo (1989) and Kos and Manea (2009) dispense with another key ingredient for the Myerson-Satterthwaite impossibility theorem—the assumption that agents have continuous value distributions—and characterize the discrete value distributions for which an ex post efficient allocation is implementable. Makowski and Mezzetti (1993) argue that the impossibility theorem is not robust to the introduction of a second buyer.

Guo and Conitzer (2008, 2009), Moulin (2009) and de Clippel et al. (2014) show that burning money and withholding goods enhance welfare in multi-unit allocation problems where agents do not have property rights.[1] Results in this literature are driven by the adoption of the stronger solution concept of dominant strategy incentive compatibility coupled with the requirement of no ex post budget deficit. By contrast, our results show that burning money and withholding goods cannot improve welfare in the Bayesian framework for bilateral trade. Remarkably, Drexl and Kleiner (2015) prove that in the absence of property rights, ex post efficient allocations are implementable in the Bayesian framework when agents have regular value distributions, closing the door on any potential benefits from withholding.[2] Our results regarding the negative welfare effects of reducing an agent's valuation in allocation problems mirror the analysis of damage for the buyer in bilateral trade. Therefore, the ownership structure in the bilateral trade setting generates fundamental asymmetries between the two traders that are not encountered in allocation problems without property rights.

The effects of withholding and damaging goods have been considered in other strategic environments by Postlewaite (1979), Deneckere and McAfee (1996), Hart and Reny (2015) and Condorelli and Szentes (2020).

The rest of the paper is organized as follows. In the next section, we set up the model with discrete types. Sections 3 and 4 present the results for money burning and withholding the good, respectively. In Sections 5 and 6, we analyze the effects of damaging the good for the buyer and for the seller, respectively. Section 7 extends our results to the continuous-type setting. In Section 9, we discuss mechanisms for allocation problems without initial ownership. Section 10 provides concluding remarks. Proofs are relegated to the Appendix.

2. Framework

We first consider a discrete-value version of the bilateral trade problem with incomplete information modeled by Myerson and Satterthwaite (1983), in which a *buyer* is interested in acquiring an indivisible *good* that a *seller* owns. The *valuations* of the buyer and of the seller for the good are independently distributed random variables with *probability mass functions* p_b and p_s, respectively, which have corresponding *supports* $V_b = \{b_1, b_2, \ldots, b_m\}$ and $V_s = \{s_1, s_2, \ldots, s_n\}$, where $0 \le b_1 < b_2 < \ldots < b_m$ and $0 \le s_1 < s_2 < \ldots < s_n$. Each trader is privately informed about his or her own value for the good (the trader's *type*) and believes that the other trader's value is a random variable drawn from the specified distribution. The two traders are risk neutral and have additively separable utility functions for money and the good.

The mechanism designer needs to specify a game that determines the probability of allocating the good to each of the two traders and the monetary transfers sent or received by the traders. The objective of the mechanism designer is to maximize expected total welfare in some Bayesian Nash equilibrium of the game. By the revelation principle, we can focus the analysis on direct mechanisms without loss of generality. In a *direct mechanism*, the traders simultaneously report their values, and the outcome is determined by four functions (x_b, x_s, t_b, t_s) with $x_b, x_s : V_b \times V_s \to [0, 1]$ and $t_b, t_s : V_b \times V_s \to \mathbb{R}$. If the buyer reports value b_i and the seller reports value s_j, then $x_b(b_i, s_j)$ and $x_s(b_i, s_j)$ are the probabilities with which the good is *allocated* to the buyer and to the seller, respectively, and $t_b(b_i, s_j)$ is the *monetary transfer sent* by the buyer, whereas $t_s(b_i, s_j)$ is the *monetary transfer received* by the seller.

This formulation departs from the Myerson-Satterthwaite model by allowing the mechanism designer to withhold both money and the good: for a profile of reports (b_i, s_j), $t_b(b_i, s_j) - t_s(b_i, s_j)$ is the amount of money withheld (or "burned"), and $1 - x_b(b_i, s_j) - x_s(b_i, s_j)$ is the probability that the good is withheld (or destroyed) by the mechanism designer. The mechanism designer does not bring any resources to the market, and *feasibility* requires that $t_b(b_i, s_j) - t_s(b_i, s_j) \ge 0$ and $x_b(b_i, s_j) + x_s(b_i, s_j) \le 1$.

A direct mechanism is (Bayesian) *incentive compatible* if honest reporting of values forms a Bayesian Nash equilibrium of the game between the buyer and the seller induced by the mechanism. A mechanism is (interim) *individually rational* if each trader type obtains non-negative expected utility gains from participating in the mechanism. We say that a mechanism (x_b, x_s, t_b, t_s) is *implementable* if it is feasible, individually rational, and incentive compatible. An allocation (x_b, x_s) is *implementable* if there exist transfer functions (t_b, t_s) such that (x_b, x_s, t_b, t_s) is an implementable mechanism. The designer's

[1] Other related contributions include Hartline and Roughgarden (2008) and Long et al. (2017).

[2] Green and Laffont (1979) showed that ex post efficiency cannot be achieved under dominant strategy implementation subject to ex post budget balance. Drexl and Kleiner note that Green and Laffont's conclusion is reversed if the budget balance constraint is relaxed from the ex post to the ex ante stage. Drexl and Kleiner also cogently argue that the results of Manelli and Vincent (2010) and Gershkov et al. (2013) regarding the equivalence of Bayesian and dominant strategy implementation in general classes of mechanism design problems do not apply under the additional desideratum of no ex post budget deficit.

objective is to specify an implementable mechanism that maximizes *total welfare*—the sum of expected utility for the buyer and the seller.

For a mechanism (x_b, x_s, t_b, t_s), let

$$\bar{x}_b(b_i) = \sum_{j=1}^{n} p_s(s_j) x_b(b_i, s_j) \ \& \ \bar{t}_b(b_i) = \sum_{j=1}^{n} p_s(s_j) t_b(b_i, s_j)$$

denote the probability that the buyer of type b_i *receives* the good and the expected payment type b_i *makes* respectively, and let

$$\bar{x}_s(s_j) = \sum_{i=1}^{m} p_b(b_i) x_s(b_i, s_j) \ \& \ \bar{t}_s(s_j) = \sum_{i=1}^{m} p_b(b_i) t_s(b_i, s_j)$$

denote the probability that the seller of type s_j *keeps* the good and the expected payment type s_j *receives*, respectively. An *optimal implementable mechanism* (x_b, x_s, t_b, t_s) solves the following linear program:[3]

$$\begin{aligned}
&\max \sum_{i=1}^{m} p_b(b_i)(\bar{x}_b(b_i)b_i - \bar{t}_b(b_i)) + \sum_{j=1}^{n} p_s(s_j)(\bar{x}_s(s_j)s_j + \bar{t}_s(s_j)) \\
&\text{s.t. } IR_{b_i} : \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) \geq 0, \forall i = \overline{1,m} \\
&\quad IR_{s_j} : \bar{x}_s(s_j)s_j + \bar{t}_s(s_j) \geq s_j, \forall j = \overline{1,n} \\
&\quad IC_{b_i \to b_k} : \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) \geq \bar{x}_b(b_k)b_i - \bar{t}_b(b_k), \forall i,k = \overline{1,m} \\
&\quad IC_{s_j \to s_k} : \bar{t}_s(s_j) - (1 - \bar{x}_s(s_j))s_j \geq \bar{t}_s(s_k) - (1 - \bar{x}_s(s_k))s_j, \forall j,k = \overline{1,n} \\
&\quad FT_{b_i,s_j} : t_b(b_i, s_j) - t_s(b_i, s_j) \geq 0, \forall i = \overline{1,m}, j = \overline{1,n} \\
&\quad FX_{b_i,s_j} : x_b(b_i, s_j) + x_s(b_i, s_j) \leq 1, \forall i = \overline{1,m}, j = \overline{1,n}.
\end{aligned}$$

The seller's individual rationality constraint IR_{s_j} guarantees that by participating in the mechanism, a seller of type s_j obtains utility at least s_j, reflecting the seller's initial ownership of the good (and option to walk away from the mechanism and hold on to the good).

Myerson and Satterthwaite (1983) assume that the mechanism designer cannot withhold either money or the good, which corresponds to imposing the feasibility constraints FT_{b_i,s_j} and FX_{b_i,s_j} with equality for all i and j. This assumption restricts the set of mechanisms the designer can implement. The next two sections investigate whether giving the mechanism designer the freedom to withhold money or the good can enhance welfare.

3. Money burning

We first show that withholding monetary transfers ("burning money") is never optimal.

Proposition 1. *Optimal implementable mechanisms never withhold monetary transfers.*

The proof of this and subsequent results are presented in the Appendix. The argument is straightforward. If an implementable mechanism involves money withholding for a pair of reports (b_i, s_j), we can increase the transfer to seller type s_j to match the transfer from b_i when the buyer reports type b_i, and reduce the transfers s_j receives from all buyer types uniformly and then credit them to the buyer, so that s_j enjoys the same expected outcomes following the perturbation. The perturbed mechanism preserves individual rationality and incentive compatibility, and increases the expected utility of the buyer without affecting the expected utility of the seller.

4. Withholding the good

The previous section shows that in solving the linear program for the optimal mechanism, we can assume that the FT_{b_i,s_j} constraints hold with equality, i.e., $t_b(b_i, s_j) = t_s(b_i, s_j)$ for all (b_i, s_j). In what follows, we restrict attention to mechanisms with this property, and use a single function t to describe monetary transfers; notation for a mechanism simplifies to (x_b, x_s, t) with the understanding that $t_b(b_i, s_j) = t_s(b_i, s_j) = t(b_i, s_j)$.

The next result shows that withholding the good cannot enhance welfare under the optimal mechanism.

[3] The notation $\overline{a,b}$ stands for the sequence of integers in the interval $[a, b]$.

Proposition 2. *There exists an optimal implementable mechanism that withholds the good with probability zero. If $s_1 > 0$, then every optimal implementable mechanism withholds the good with probability zero.*

The following lemmata collect standard insights from mechanism design about binding "local" incentive constraints and monotonicity properties of the allocation needed for the proof (some appear in Lovejoy, 2006).

Lemma 1. *A mechanism (x_b, x_s, t) is implementable if and only if it satisfies IR_{b_1}, IR_{s_n}, $IC_{b_i \to b_{i+1}}$, $IC_{b_{i+1} \to b_i}$, $IC_{s_j \to s_{j+1}}$ and $IC_{s_{j+1} \to s_j}$ for $i = \overline{1, m-1}$ and $j = \overline{1, n-1}$. If (x_b, x_s, t) is implementable, then $\bar{x}_b(b_{i+1}) \geq \bar{x}_b(b_i)$ and $\bar{x}_s(s_{j+1}) \geq \bar{x}_s(s_j)$ for $i = \overline{1, m-1}$ and $j = \overline{1, n-1}$. For any mechanism (x_b, x_s, t), if $\bar{x}_b(b_{i+1}) > \bar{x}_b(b_i)$ and $IC_{b_{i+1} \to b_i}$ holds with equality, then $IC_{b_i \to b_{i+1}}$ is satisfied with strict inequality. Similarly, if $\bar{x}_s(s_{j+1}) > \bar{x}_s(s_j)$ and $IC_{s_j \to s_{j+1}}$ holds with equality, then $IC_{s_{j+1} \to s_j}$ is satisfied with strict inequality.*

Lemma 2. *For every implementable mechanism (x_b, x_s, t), there exists a transfer specification t' such that (x_b, x_s, t') is implementable, and IR_{b_1}, $IC_{b_i \to b_{i-1}}$ and $IC_{s_j \to s_{j+1}}$ hold with equality under (x_b, x_s, t') for $i = \overline{2, m}$ and $j = \overline{1, n-1}$.*

We sketch the proof for the second part of Proposition 2 here (the first part follows from similar ideas and a compactness argument). Suppose that $s_1 > 0$, and consider an implementable mechanism (x_b, x_s, t'') that withholds the good with positive probability for a profile of reports (b_i, s_j), i.e., $x_b(b_i, s_j) + x_s(b_i, s_j) < 1$. Lemma 2 implies the existence of a transfer function t that implements the same allocation (x_b, x_s) such that (x_b, x_s, t) satisfies the constraints listed in the lemma with equality. Since neither mechanism withholds money, (x_b, x_s, t) generates the same total welfare as (x_b, x_s, t''), so (x_b, x_s, t'') is optimal if and only if (x_b, x_s, t) is optimal. To increase the welfare generated by (x_b, x_s, t), we contemplate a perturbation of the allocation whereby the seller of type s_j keeps the good with higher probability in the event agents report types (b_i, s_j): $x'_s(b_i, s_j) = x_s(b_i, s_j) + \varepsilon$.

If $\bar{x}_s(s_j) < \bar{x}_s(s_{j+1})$ and ε is sufficiently small, the monotonicity of $\bar{x}_s$ implies that $\bar{x}'_s$ is monotonic, which per Lemma 1 is a necessary condition for the implementation of the allocation (x_b, x'_s). To provide incentives supporting the new allocation (x_b, x'_s), we modify transfers to type s_j uniformly so that s_j receives the same utility in the perturbed mechanism: $t'(b_k, s_j) = t(b_k, s_j) - p_b(b_i)\varepsilon s_j$ for all k. $IC_{s_{j+1} \to s_j}$ is the main constraint from Lemma 1 we need to check in order to establish that (x_b, x'_s, t') is an implementable mechanism. Indeed, the perturbation reduces payments to a type s_j by $p_b(b_i)\varepsilon s_j$ to account for the extra probability $p_b(b_i)\varepsilon$ with which this type is allocated the good. This marginal change is also attractive to type s_{j+1} for whom the increase in the allocation probability is worth $p_b(b_i)\varepsilon s_{j+1}$, which is greater than the implicit cost of $p_b(b_i)\varepsilon s_j$. However, the condition $\bar{x}_s(s_j) < \bar{x}_s(s_{j+1})$ and the assumption that (x_b, x_s, t) satisfies $IC_{s_j \to s_{j+1}}$ with equality, along with Lemma 1, imply that (x_b, x_s, t) satisfies $IC_{s_{j+1} \to s_j}$ with strict inequality. By continuity, (x_b, x'_s, t') must also satisfy $IC_{s_{j+1} \to s_j}$ for small ε. The perturbation increases welfare by $p_b(b_i)p_s(s_j)\varepsilon s_j > 0$, which means that (x_b, x_s, t) is suboptimal.

The case $\bar{x}_s(s_j) = \bar{x}_s(s_{j+1})$ requires a more extensive sequence of perturbations since the monotonicity condition necessary for implementability from Lemma 1 cannot be maintained if we increase $\bar{x}_s(s_j)$ without altering $\bar{x}_s(s_{j+1})$. However, in seeking a welfare improvement over the mechanism (x_b, x_s, t), we still take advantage of the slack in the constraint FX_{b_i, s_j} by increasing the probability that type s_j keeps the good to $x'_s(b_i, s_j) = x_s(b_i, s_j) + \varepsilon$ for some small ε. For this change to have a neutral effect on $\bar{x}_s(s_j)$, we decrease $x_s(b_{i'}, s_j)$ and increase $x_b(b_{i'}, s_j)$ by $\varepsilon p_b(b_i)/p_b(b_{i'})$ for some other buyer type $b_{i'}$. We can then neutralize the effect of this perturbation on type $b_{i'}$ by decreasing the probability that $b_{i'}$ receives the good from some other seller type $s_{j'}$. Finally, the increase in the probability that $s_{j'}$ keeps the good can be implemented without a further change in the allocation of $s_{j'}$ using the idea from the first step of the proof if $\bar{x}_s(s_{j'}) < \bar{x}_s(s_{j'+1})$. In the proof, we choose $s_{j'}$ to be the highest seller type such that $\bar{x}_s(s_{j'}) = \bar{x}_s(s_j)$, and demonstrate the existence of a buyer type $b_{i'}$ such that all perturbations generate allocation probabilities in $[0, 1]$. We then adjust transfers for seller type $s_{j'}$ as in the first step of the proof to preserve the structure of incentives for the seller. We show that the perturbed mechanism boosts the welfare of (x_b, x_s, t) by $p_b(b_i)p_s(s_j)\varepsilon s_{j'} > 0$, proving that (x_b, x_s, t) is not optimal.

Interestingly, an analogous argument does not go through if $\bar{x}_b(b_i) < \bar{x}_b(b_{i+1})$, and we perturb the mechanism (x_b, x_s, t) to allocate the good with ε additional probability to the buyer—rather than to the seller—in the event agents report types (b_i, s_j), i.e., $x'_b(b_i, s_j) = x_b(b_i, s_j) + \varepsilon$. Indeed, if $IC_{b_{i+1} \to b_i}$ and $IC_{b_i \to b_{i-1}}$ are binding under (x_b, x_s, t), then maintaining these constraints under the perturbed mechanism (x'_b, x'_s, t') requires that $p_s(s_j)\varepsilon b_{i+1} \leq \bar{t}'_b(b_i) - \bar{t}_b(b_i) \leq p_s(s_j)\varepsilon b_i$, which is impossible because $p_s(s_j) > 0$ and $b_{i+1} > b_i$. This explains the buyer-seller asymmetry in the statement of Proposition 2. The forthcoming analysis of damaging the good for the buyer and for the seller reveals other asymmetries in the interaction between the allocation and incentives for the two agents.

5. Damaging the good for the buyer

In this section and the next, we examine whether damaging the good for the buyer or for the seller can enhance welfare. Specifically, we assume that the mechanism designer can reduce the value of either the buyer or the seller for the good by shifting the corresponding value distribution down in the sense of first-order stochastic dominance. The shift does not have to be uniform across agent types. If the good has multiple components (or is offered as a bundle) and there is heterogeneity in how agent types value each component, then the mechanism designer may "damage" the good by prohibiting the sale or regulating the use of certain components with varying effects on different types. Uniform value reduction can also be accomplished by imposing a property tax.

Since Propositions 1 and 2 show that withholding money and the good cannot improve welfare, we revert to the Myerson-Satterthwaite formulation of the optimal mechanism in which the constraints FT_{b_i,s_j} and FX_{b_i,s_j} are imposed with equality. Following the notation from the previous section, the *transfer* function $t: V_b \times V_s \to \mathbb{R}$ specifies a payment from the buyer to the seller. We *update notation* for allocations to reflect the fact that withholding the good is ruled out: the *allocation* is now determined by a single function $x: V_b \times V_s \to [0,1]$ that describes the *probability of trade* between the buyer and the seller. In the *mechanism* (x,t), when the buyer reports type b_i and the seller reports type s_j, the two agents trade the good with probability $x(b_i,s_j)$ in exchange for a payment $t(b_i,s_j)$.

Let

$$\bar{x}_b(b_i) = \sum_{j=1}^{n} p_s(s_j)x(b_i,s_j) \;\&\; \bar{t}_b(b_i) = \sum_{j=1}^{n} p_s(s_j)t(b_i,s_j)$$

denote the probability that the buyer of type b_i *buys* the good and the expected payment type b_i *sends* to the seller, respectively, and let

$$\bar{x}_s(s_j) = \sum_{i=1}^{m} p_b(b_i)x(b_i,s_j) \;\&\; \bar{t}_s(s_j) = \sum_{i=1}^{m} p_b(b_i)t(b_i,s_j)$$

denote the probability that the seller of type s_j *sells* the good and the expected payment type s_j *receives* from the buyer, respectively (note the *change in notation*: unlike in the previous sections, $\bar{x}_s$ denotes trading probabilities for the seller here). The optimal implementable mechanism (x,t) then solves the following linear program:

$$\begin{aligned}
&\max \sum_{i=1}^{m} p_b(b_i)\bar{x}_b(b_i)b_i + \sum_{j=1}^{n} p_s(s_j)(1-\bar{x}_s(s_j))s_j \\
&\text{s.t. } IR_{b_i}: \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) \geq 0, \forall i = \overline{1,m} \\
&\quad IR_{s_j}: \bar{t}_s(s_j) - \bar{x}_s(s_j)s_j \geq 0, \forall j = \overline{1,n} \\
&\quad IC_{b_i \to b_k}: \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) \geq \bar{x}_b(b_k)b_i - \bar{t}_b(b_k), \forall i,k = \overline{1,m} \\
&\quad IC_{s_j \to s_k}: \bar{t}_s(s_j) - \bar{x}_s(s_j)s_j \geq \bar{t}_s(s_k) - \bar{x}_s(s_k)s_j, \forall j,k = \overline{1,n}.
\end{aligned}$$

We establish that damaging the good for the buyer can never improve welfare.

Proposition 3. *If p_b and p'_b are two value distributions for the buyer such that p_b first-order stochastically dominates p'_b, then for any seller value distribution p_s, no implementable mechanism for the pair of value distributions (p'_b, p_s) achieves greater total welfare than the optimal implementable mechanism for (p_b, p_s).*

The intuition for this result is that reducing buyer value entails less scope for allocating the good to the buyer and thus decreases potential gains from trade. For the proof, it is helpful to approach this intuition from the opposite angle: if the buyer value distribution improves from p'_b to p_b, it becomes easier to incentivize trade with buyer types whose values increase.

We can assume that the optimal mechanisms for (p_b, p_s) and (p'_b, p_s) are defined for buyer valuations in the union of the supports of p_b and p'_b since any implementable mechanism can be extended to a set of probability-zero types by specifying allocations and transfers for each such type as an optimal selection for that type from the menu of allocations and transfers prescribed by the mechanism for positive-probability types, along with the option of no trade and zero transfers. For any pair of distributions with finite supports that is ranked with respect to first-order stochastic dominance, we can reach the dominating distribution from the dominated one via a finite sequence of changes that move probability mass up between single pairs of adjacent points in the union of the two supports. Therefore, it is sufficient to establish the result for the case in which $p_b(b_i) = p'_b(b_i) - \delta$ and $p_b(b_{i+1}) = p'_b(b_{i+1}) + \delta$ for some $\delta > 0$, and $p'_b(b_k) = p_b(b_k)$ for $k \neq i, i+1$.

We show that any mechanism (x', t') that is implementable when traders' values are distributed according to (p'_b, p_s) and satisfies the constraint $IC_{b_{i+1} \to b_i}$ from Lemma 2 with equality can be perturbed to obtain an implementable mechanism (x,t) for the value distributions (p_b, p_s) that differs from (x', t') only for the following profiles of types:

$$\begin{aligned}
x(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})}x'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}x'(b_{i+1}, s_j) \\
t(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})}t'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}t'(b_{i+1}, s_j), \forall j = \overline{1,n}.
\end{aligned}$$

The perturbed mechanism (x, t) channels $\delta \bar{x}'_b(b_i)$ volume of trade from the measure δ of buyer types whose valuations improve from b_i to b_{i+1} in the shift from p'_b to p_b without affecting the expected pattern of trade for other buyer and seller types, and consequently generates $\delta \bar{x}'_b(b_i)(b_{i+1} - b_i) \geq 0$ more surplus than (x', t').

6. Damaging the good for the seller

We now consider the situation in which the mechanism designer can damage the good for the seller. It is reasonable to assume that the seller has the option to keep the undamaged good, so the mechanism designer is constrained to provide each seller type at least the utility the seller would derive from consuming the good prior to the damage. To express this participation constraint for the seller, we focus on *type-by-type* damage whereby the value of each seller type j is reduced from s_j to s'_j. Thus, the mechanism designer can shift down the initial distribution of seller values p_s with support $s_1 < s_2 < \ldots < s_n$ to a distribution p'_s with support $s'_1 < s'_2 < \ldots < s'_n$ such that $s'_j \leq s_j$ and $p_s(s_j) = p'_s(s'_j)$ for $j = \overline{1, n}$. Then, in the notation of the previous section, the *damage participation constraint* for seller type j in an implementable mechanism (x, t) following the value reduction from p_s to p'_s requires that

$$\bar{t}_s(s'_j) + (1 - \bar{x}_s(s'_j))s'_j \geq s_j.$$

Note that the standard $IR_{s'_j}$ constraint for the pair of distributions (p_b, p'_s) is weaker, with s_j being replaced by s'_j on the right-hand side of the inequality above.

In contrast to Proposition 3, the next result shows that damaging the good for the seller may enhance welfare.

Proposition 4. *Let x be an implementable allocation for a pair of value distributions (p_b, p_s). Suppose that there exist $i \leq m - 1$ and $j \leq n - 1$ such that $\bar{x}_b(b_1) = \bar{x}_b(b_i) < \bar{x}_b(b_{i+1})$ and $\bar{x}_s(s_1) = \bar{x}_s(s_j) > \bar{x}_s(s_{j+1}) > 0$. If $b_i > s_1$ and $x(b_i, s_1) < 1$, then for any sufficiently small $\varepsilon > 0$, damaging the good to reduce the valuations of seller types $1, \ldots, j$ uniformly by ε (without affecting other types) enables the mechanism designer to implement an allocation x' (while obeying the seller's damage participation constraint) that coincides with x for every pair of corresponding buyer and seller types except that*

$$x'(b_i, s_1 - \varepsilon) = x(b_i, s_1) + \varepsilon \frac{(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) \sum_{l=1}^{j} p_s(s_l)}{p_s(s_1)(b_{i+1} - b_i) \sum_{k=i+1}^{m} p_b(b_k)}.$$

It is possible that the allocation x' generates greater total welfare given the damaged seller values than x given the original seller values even in situations where x is an optimal implementable allocation for (p_b, p_s).[4]

To understand this result, note that $\bar{x}_s(s_1) = \bar{x}_s(s_j)$ implies that any mechanism implementing allocation x generates the same expected outcomes ($\bar{x}_s$ and $\bar{t}_s$) for seller types 1 through j, and damaging the good for these types decreases their incentives to mimic higher types. Indeed, higher seller types keep the good with higher probability, but keeping the good becomes less attractive for a low-type seller whose value has been reduced. The slack created in the constraints $IC_{s_{j'} \to s_{j+1}}$ by the ε reduction in valuations for seller types $j' \leq j$ can be used to decrease transfers from the buyer of type $i + 1$ and higher to the seller of type j and lower. This perturbation introduces slack in the constraint $IC_{b_{i+1} \to b_i}$, which can be leveraged to increase the probability of an ex post efficient trade between the seller with reduced valuation $s_1 - \varepsilon$ and the buyer with valuation b_i at marginal terms that make type b_i indifferent and would be attractive to type b_{i+1} if not for the decrease in transfers granted to this type under the perturbation.

However, the improvement in allocative efficiency generated by increased trading between types b_i and $s_1 - \varepsilon$ is achieved at the cost of damaging the good and thus reducing the utility of affected seller types in the event of no trade. Hence, there is a trade-off between the additional gains from trade for the pair $(b_i, s_1 - \varepsilon)$,

$$p_b(b_i) p_s(s_1)(b_i - s_1 + \varepsilon)(x'(b_i, s_1 - \varepsilon) - x(b_i, s_1)) = \varepsilon \frac{(b_i - s_1 + \varepsilon)(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) p_b(b_i) \sum_{l=1}^{j} p_s(s_l)}{(b_{i+1} - b_i) \sum_{k=i+1}^{m} p_b(b_k)}, \quad (1)$$

and the expected utility loss suffered by seller types 1 through j in the event that they do not trade the damaged good,

$$\varepsilon (1 - \bar{x}_s(s_1)) \sum_{l=1}^{j} p_s(s_l). \quad (2)$$

Both the marginal efficiency gain deriving from the perturbed allocation and the marginal loss resulting from the damage are of order ε, and the net effect of the perturbation on total welfare depends on the original allocation x and the value distributions (p_b, p_s).

[4] We can replace the hypothesis $x(b_i, s_1) < 1$ with the weaker condition $\bar{x}_s(s_1) < 1$, and adjust the conclusion to state that $\bar{x}'$ coincides with $\bar{x}$ for all corresponding buyer and seller types except that $\bar{x}'_b(b_i) = \bar{x}_b(b_i) + \varepsilon' p_s(s_1)$ and $\bar{x}'_s(s_1 - \varepsilon) = \bar{x}_s(s_1) + \varepsilon' p_b(b_i)$ for some $\varepsilon' > 0$. See the Appendix for a proof.

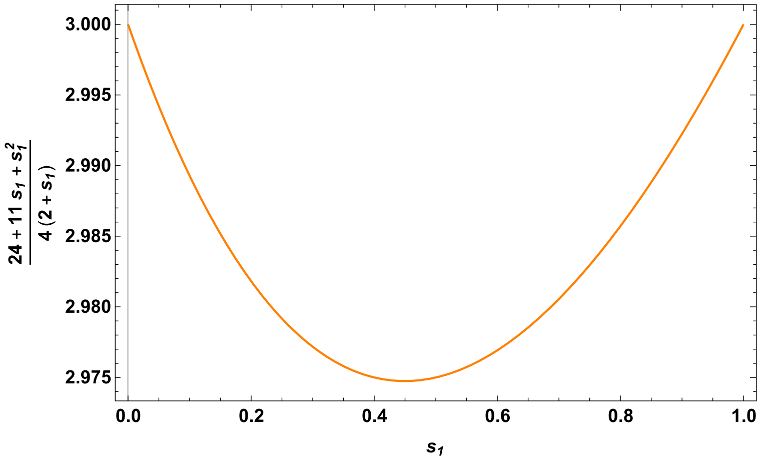

Fig. 1. Total welfare under the optimal mechanism as a function of s_1.

We illustrate this trade-off in an example showing that either effect (1) or (2) can dominate when x is the optimal implementable allocation for the original value distributions. Suppose that there are two buyer and two seller types ($m = n = 2$). The buyer's valuations are $b_1 = 1$ and $b_2 = 4$ with equal probability, while the seller's valuations are (a parameter) $s_1 \in [0, 1)$ and $s_2 = 3$ with equal probability. We assume that the designer can damage the good for the low-type seller alone, and consider the welfare consequences of reducing s_1.

In the Appendix, we show that the optimal mechanism implements the following allocation:

$$x(b_1, s_1) = \frac{2}{2 + s_1},\ x(b_1, s_2) = 0,\ x(b_2, s_1) = 1,\ x(b_2, s_2) = 1. \tag{3}$$

The optimal mechanism allocates the good efficiently with higher probability as s_1 decreases to 0, reaching ex post efficiency for $s_1 = 0$.[5] As implied by Proposition 4, making the good less valuable for the low-type seller s_1 reduces her incentive to mimic the high type s_2, who trades less in the optimal mechanism. The resulting slack in the constraint $IC_{s_1 \to s_2}$ is leveraged to increase the probability of trade between types b_1 and s_1.

The optimal implementable allocation x identified in (3) generates a total welfare of

$$\frac{24 + 11s_1 + s_1^2}{4(2 + s_1)},$$

which is decreasing for $s_1 \in [0, \sqrt{6} - 2]$ and increasing for $s_1 \in [\sqrt{6} - 2, 1)$ as seen in Fig. 1. At $s_1 = \sqrt{6} - 2 \approx 0.449$, the marginal improvement in gains from trade quantified by formula (1) resulting from a small decrease in s_1 exactly offsets the marginal utility loss described by expression (2) that seller type s_1 suffers in the event she retains the damaged good. For $s_1 < \sqrt{6} - 2$, the gain is greater than the loss, while for $s_1 > \sqrt{6} - 2$ the opposite is true. For a concrete computation, when s_1 takes values 0 and 0.4, the corresponding optimal implementable mechanisms achieve total welfare 3 and 2.975, respectively. Therefore, reducing the value of the low-type seller from 0.4 to 0 enhances welfare under the optimal mechanism.

The seller's damage participation constraint is satisfied in this example when the designer damages the good for the low-type seller and implements the optimal mechanism given the seller's reduced valuation. Indeed, a seller with any initial valuation $s_1 \in (0, 1)$ would consent to the damage and then participate in the optimal mechanism with a reduced valuation $s_1' \in [0, s_1)$ because trading in the optimal mechanism with value s_1' yields an expected payoff of at least 3/2, which exceeds the seller's value s_1 for the undamaged good. To see this, note that the individual rationality constraint IR_{s_2} and the incentive compatibility constraint $IC_{s_1' \to s_2}$ hold with equality under the optimal mechanism for the reduced value s_1'. Since seller type s_2 trades the good with probability 1/2 under this mechanism, she receives an expected payment of $1/2 \times s_2 = 3/2$. Then, seller type s_1' expects a utility of $1/2 \times s_1' + 3/2 \geq 3/2$ under the mechanism.

Proposition 4 presumes that the mechanism designer is able to target the damage of the good at seller types with low valuations. Suppose instead that damage must hurt the seller type with the highest valuation most severely. Specifically, damage is restricted to shifting down the seller value distribution p_s with support $s_1 < s_2 < \ldots < s_n$ to distributions p_s' with support $s_1' < s_2' < \ldots < s_n'$ such that $0 \leq s_j - s_j' \leq s_n - s_n'$ and $p_s'(s_j') = p_s(s_j)$ for $j = \overline{1, n}$. In this case, we say that p_s' *reflects damage at the top relative to* p_s. We show that damage at the top for the seller cannot raise expected welfare.

[5] Myerson and Satterthwaite (1983) note that their result about the impossibility of implementing the ex post efficient allocation for continuous value distributions does not extend to the case of discrete distributions. This example with $s_1 = 0$ is an instance where their impossibility result does not apply.

Proposition 5. *If p_s' is a seller value distribution that reflects damage at the top relative to p_s, then for any buyer value distribution p_b, there is no implementable mechanism for the pair of value distributions (p_b, p_s') that obeys the seller's damage participation constraint for the value reduction from p_s to p_s' and achieves greater total welfare than the optimal implementable mechanism for (p_b, p_s).*[6]

To develop intuition for this result, recall from the discussion of Proposition 4 that reducing the valuation of a seller type that is not the highest introduces slack in the incentives for that type not to report higher values if the same allocation is to be implemented; this decreases the information rent needed to compensate that type relative to the next higher type. However, this potentially positive welfare effect is offset by the direct loss created by the damage as well as by the increased transfers required to meet the seller's damage participation constraint. In particular, the damage participation constraint for the seller with the highest valuation is more restrictive than the post-damage individual rationality constraint for this type; a higher expected transfer is required for this type to implement the same allocation. The proof of Proposition 5 shows that under damage at the top, the additional transfer imposed by the damage participation constraint for the seller type with the highest valuation is greater than the sum of transfers saved on lower types as a result of the diminished information rents dispensed to these types after the damage.

7. Results for continuous types

In this section, we present versions of our results for the original Myerson-Satterthwaite model, in which buyer and seller valuations have continuous distributions. Assume that the valuations of the buyer and the seller have cumulative distribution functions F_b and F_s with continuous densities $f_b = F_b'$ and $f_s = F_s'$ that are positive at every point in their respective supports $[\underline{b}, \bar{b}]$ and $[\underline{s}, \bar{s}]$. Recall that the virtual values of the buyer of type b and the seller of type s are defined by

$$\psi_b^{F_b}(b) = b - \frac{1 - F_b(b)}{f_b(b)} \text{ and } \psi_s^{F_s}(s) = s + \frac{F_s(s)}{f_s(s)}, \text{ respectively.}$$

A buyer distribution F_b or a seller distribution F_s is said to be *regular* if its respective virtual value function $\psi_b^{F_b}$ or $\psi_s^{F_s}$ is strictly increasing.

In the continuous-type setting, a direct mechanism (x_b, x_s, t_b, t_s) that permits withholding is described by allocation functions $x_b, x_s : [\underline{b}, \bar{b}] \times [\underline{s}, \bar{s}] \to [0, 1]$ and transfer functions $t_b, t_s : [\underline{b}, \bar{b}] \times [\underline{s}, \bar{s}] \to \mathbb{R}$ that are Riemann integrable. Constraints for an implementable mechanism are specified analogously to the discrete-type setting (with integrals replacing summations).

The first result for the continuous-type setting establishes the counterpart of Proposition 1—money burning is never optimal. The proof relies on a perturbation similar to the one used to prove Proposition 1.

Proposition 6. *Suppose that (x_b, x_s, t_b, t_s) is an optimal implementable mechanism for a pair of value distributions (F_b, F_s). Then,*

$$\int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(b, s) - t_s(b, s)) f_b(b) f_s(s)\, db\, ds = 0.$$

Given this result, in solving for optimal mechanisms we can now assume that $t_b(b, s) = t_s(b, s)$ for all pairs of types (b, s). We simplify notation for a mechanism to (x_b, x_s, t) with the understanding that $t_b(b, s) = t_s(b, s) = t(b, s)$.

We next prove the analog of Proposition 2—withholding the good with positive probability is not optimal—under the assumption that buyer and seller value distributions are regular. The proof shows that the optimal mechanism characterized by Myerson and Satterthwaite in the case of regular value distributions continues to be optimal when the designer is given the extra flexibility to withhold the good.

Proposition 7. *Under any optimal implementable mechanism (x_b, x_s, t) for a pair of regular value distributions (F_b, F_s), the set of pairs of types (b, s) for which $x_b(b, s) + x_s(b, s) \neq 1$ has probability zero under the product distribution $F_b \times F_s$.*

In light of Proposition 7, the rest of this section assumes that $x_b(b, s) + x_s(b, s) = 1$ for all pairs (b, s). A mechanism is now simply denoted by (x, t), with allocation functions given by $x_b(b, s) = x(b, s)$ and $x_s(b, s) = 1 - x(b, s)$. This is the

[6] If the mechanism designer could damage the good without the seller's consent—i.e., has to abide by the individual rationality constraint with respect to the seller's reduced valuation, but not by the more restrictive damage participation constraint—then a modification of the example above demonstrates that damage at the top may increase welfare under the optimal mechanism. This is possible even when the mechanism designer is subject to an ex ante damage participation constraint requiring that the seller's expected payoff in the post-damage mechanism is not smaller than the seller's expected value for the undamaged good. See the Appendix for a discussion.

universe of mechanisms considered by Myerson and Satterthwaite from the outset of their analysis. However, we do not maintain the hypothesis of Proposition 7 that value distributions are regular.

The next result extends Proposition 3 to the continuous-type setting under the additional assumption that the damage for each buyer type results in a reduction in corresponding virtual values. Specifically, we interpret damaging the good for the buyer as a first-order stochastic dominance shift from F_b to G_b whereby the value of the buyer of type b is reduced from b to $\alpha(b) := G_b^{-1} \circ F_b(b)$, and require that the virtual value of type b under the original distribution F_b is greater than or equal to the virtual value of the corresponding type $\alpha(b)$ under the post-damage distribution G_b: $\psi_b^{F_b}(b) \geq \psi_b^{G_b}(\alpha(b))$. Since $G_b(\alpha(b)) = F_b(b)$ and $g_b(\alpha(b))\alpha'(b) = f_b(b)$, this sufficient condition is equivalent to

$$b - \alpha(b) \geq (1 - \alpha'(b))\frac{1 - F_b(b)}{f_b(b)}. \tag{4}$$

The inequality above can be rewritten as

$$\frac{f_b(b)}{1 - F_b(b)} \geq \frac{1 - \alpha'(b)}{b - \alpha(b)} \iff (\log[(b - \alpha(b))(1 - F_b(b))])' \leq 0,$$

which is equivalent to $(b - \alpha(b))(1 - F_b(b))$ being a decreasing function of b.

The proof combines Myerson and Satterthwaite's feasibility condition for implementing monotonic allocations with the intuition behind Proposition 3 that an implementable mechanism under (G_b, F_s) can be modified to obtain an implementable mechanism under (F_b, F_s) by redirecting trade with type $\alpha(b)$ under G_b to type b under F_b in a way that does not affect trading probabilities for individual seller types.

Proposition 8. *Let F_b and G_b be distributions of buyer values such that F_b first-order stochastically dominates G_b, and denote $\alpha = G_b^{-1} \circ F_b$. Suppose that $\psi_b^{F_b}(b) \geq \psi_b^{G_b}(\alpha(b))$ for all b in the support of F_b. Then, for every seller value distribution F_s, the total welfare under an optimal implementable mechanism for the pair of value distributions (G_b, F_s) does not exceed the one for (F_b, F_s).*

The final result for the continuous-type setting extends Proposition 5 under a different version of the hypothesis that no seller type is hurt by the damage more than the type with the highest valuation. The new hypothesis presumes that damage for seller type s whose value is reduced to $\alpha(s)$ via a first-order stochastic dominance shift from F_s to G_s is reflected by the difference in virtual values of type s under the distribution F_s and the corresponding type $\alpha(s)$ under the distribution G_s. The hypothesis requires that the reduction in virtual values $\psi_s^{F_s}(s) - \psi_s^{G_s}(\alpha(s))$ for every type s should not exceed the reduction in actual values $\bar{s} - \alpha(\bar{s})$ for the highest type $\bar{s}$.[7] As in the discrete-type setting, post-damage mechanisms must obey the seller's damage participation constraint, which requires that for every s in the support of F_s, the seller type with reduced valuation $\alpha(s)$ receives an expected utility of at least s from participating in the mechanism after the damage.

Proposition 9. *Let F_s and G_s be distributions of seller values such that F_s first-order stochastically dominates G_s, and denote $\alpha = G_s^{-1} \circ F_s$. Suppose that $\psi_s^{F_s}(s) - \psi_s^{G_s}(\alpha(s)) \leq \bar{s} - \alpha(\bar{s})$ for all s in the support of F_s, where $\bar{s}$ denotes the maximum of the support of F_s. Then, there is no implementable mechanism for the pair of value distributions (F_b, G_s) that obeys the seller's damage participation constraint for the reduction in values from F_s to G_s and achieves greater total welfare than the optimal mechanism for (F_b, F_s).*

8. Comparisons of methods and results for discrete and continuous value distributions

The proofs of Propositions 7 through 9 rely on applying revenue equivalence for the continuous-type setting (Myerson, 1981) and adjusting the argument for Theorem 1 in Myerson and Satterthwaite (1983) to show that an allocation (x_b, x_s) (that potentially involves withholding the good) is implementable if and only if $\bar{x}_b$ and $\bar{x}_s$ are increasing functions and the following inequality holds:[8]

$$\int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\bar{s}} \left(x_b(b,s)\psi_b^{F_b}(b) - (1 - x_s(b,s))\psi_s^{F_s}(s)\right) f_b(b) f_s(s)\, ds\, db \geq 0.$$

This result allows us to verify whether allocations are implementable without specifying supporting transfers.

Revenue equivalence also holds for mechanisms that satisfy the local incentive compatibility constraints from Lemma 2 (downward for buyer types, and upward for seller types) with equality in the discrete-type setting. As in the continuous-type setting, it can be shown that an allocation (x_b, x_s) is implementable if and only if $\bar{x}_b$ and $\bar{x}_s$ are increasing functions and the inequality aggregating these constraints holds:

[7] For $s = \bar{s}$, the hypothesis requires that $\alpha'(\bar{s}) \leq 1$.

[8] Recall that in our setting $\bar{x}_s(s)$ represents the probability that seller type s keeps the good, and not the probability that type s trades the good, which is the primitive variable in the analysis of Myerson and Satterthwaite; the two variables have opposite monotonicity.

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(x_b(b_i,s_j)\psi_b^{p_b}(b_i)-(1-x_s(b_i,s_j))\psi_s^{p_s}(s_j)\right)p_b(b_i)p_s(s_j)\geq 0.$$

Here $\psi_b^{p_b}$ and $\psi_s^{p_s}$ denote the discrete-type analogues of Myerson's (1981) virtual values (e.g., Lovejoy, 2006):

$$\psi_b^{p_b}(b_i)=b_i-(b_{i+1}-b_i)\frac{p_b(b_{i+1})+\ldots+p_b(b_m)}{p_b(b_i)} \text{ and } \psi_s^{p_s}(s_j)=s_j+(s_j-s_{j-1})\frac{p_s(s_1)+\ldots+p_s(s_{j-1})}{p_s(s_j)}.$$

Versions of this "folk theorem" appear in Kos and Manea (2009) and Schottmuller (2023). One can use this result to verify that the various perturbations of allocations underlying our arguments for Propositions 2 and 3 are implementable without specifying the corresponding transfers.[9] Nevertheless, we took an approach that explicitly defines the accompanying transfer perturbations in order to develop intuition into how slack in the local incentive compatibility constraints from Lemma 2 can be leveraged towards welfare improvement.

Embedded in the characterization of implementable allocations without reference to supporting transfers is the equivalence between ex ante and ex post budget balance constraints in Bayesian mechanisms with independent types. This equivalence is showcased in the early constructions of Arrow (1979), d'Aspremont and Gerard-Varet (1979) and Myerson and Satterthwaite (1983), and has been established in general environments by Borgers and Norman (2009). Proposition 2 in the latter study applied to the bilateral trade problem implies that for any implementable mechanism that runs an expected budget surplus $\Delta \geq 0$, there exists an implementable mechanism that implements the same allocation and runs an ex post budget surplus Δ for every profile of types. Our Propositions 1 and 6 can then be viewed as corollaries of this result.

Propositions 7 through 9 invoke hypotheses related to virtual values that were not needed for their discrete-type counterparts. These hypotheses would appear as restrictive in the context of discrete value distributions as they are for continuous ones; in this sense, our results for the discrete-type setting are stronger. Whether the results for continuous types continue to hold if we drop the hypotheses pertaining to virtual values remains a question for future research. One approach that might be useful in extending our results to all continuous distributions would be to develop convergent upper and lower bounds on total welfare under the optimal mechanism for any pair of continuous distributions via discrete grid approximations, and derive Propositions 7, 8 and 9 from Propositions 2, 3 and 5, respectively.[10] The underlying convergence argument would entail showing that the value of the linear program associated with the optimal mechanism for discrete value distributions varies continuously with the distribution (within the class of discrete distributions employed to derive upper and lower bounds). This is a separate hurdle since the optimal value of a linear program is not always a continuous function of coefficients in the constraints (see Gale, 1960 (p. 95), Williams, 1963 and Wets, 1985).

9. Allocation mechanisms without property rights

We extend our analysis to the model of social choice rules formulated by Wolitzky (2016). A set N of $n \geq 1$ agents face a set of alternatives $X \subset [0,\infty)^n$. The type of each agent i is drawn independently from a set of nonnegative numbers V_i, which is either finite or a closed interval. Agents have a common prior, and agent i is privately informed about his type v_i. The set of type profiles is denoted by $V=\prod_{i\in N} V_i$. If alternative $x \in X$ is implemented, then an agent i of type v_i who makes a payment t_i gets payoff $x_i v_i - t_i$. Special instances of this framework include allocation problems for multiple units of an indivisible good for which agents have unit demand (in this case, x_i is the probability that agent i receives a unit) and allocation problems for a divisible good (in this case, x_i is the quantity agent i receives).[11]

In this environment, a direct mechanism specifies an allocation function $x: V \to X$ and a transfer function $t: V \to \mathbb{R}^n$. The IC, IR and FT constraints and total welfare for Bayesian implementable mechanisms are defined analogously to those for bilateral trade (the FX constraint is encapsulated in the requirement that the range of x is X).[12]

We assume that the set of alternatives X is convex and compact. Convexity captures the ability of the mechanism designer to randomize over alternatives, and compactness ensures the existence of optimal implementable mechanisms.

The analog of Propositions 1 and 6 in this new environment is that for every optimal implementable mechanism (x,t), all FT constraints hold with equality, that is, $\sum_{i\in N} t_i(v)=0$ for all $v \in V$ (with probability 1). Adapting the proofs for both the discrete- and the continuous-type settings requires primarily notational adjustments.

Proposition 3 also admits a straightforward extension to the discrete-type version of this environment: a reduction in any agent's valuation in terms of first-order stochastic dominance cannot enhance welfare under the optimal implementable

[9] This approach is not directly applicable to Propositions 4 and 5 due to additional restrictions on transfers stemming from the damage participation constraint for the seller.

[10] Madarasz and Prat (2017) carry on a related exercise in a screening problem with a single agent, and construct a mechanism in which the agent's optimal decisions may respond discontinuously to approximations of his type, but affect the principal's profit continuously. In our setting, perturbations in the mechanism affect the optimal reports of both the buyer and the seller, and the challenge is to maintain the continuity of the allocation resulting from this interaction.

[11] This framework does *not* formally nest our bilateral trade setting because it rules out property rights. Wolitzky applies it to bilateral trade by assuming that the seller does not hold the good, but instead produces it ex post at a privately known cost only when the mechanism prescribes that trade take place.

[12] In contrast to the Bayesian approach we maintain in this analysis, Wolitzky studies the case of agents with maxmin preferences.

mechanism. As in the case of bilateral trade, the main step in proving this claim relies on the assumed convexity of the set X of feasible alternatives. Similarly, Proposition 8 extends to the continuous-type version of the current environment under the same hypothesis regarding the effect of damage on the agent's virtual values.[13]

To draw parallels with Propositions 2 and 7 in this environment, consider an instance in which the set of alternatives X represents a multi-unit allocation problem in which agents have unit demand. In this setting, for both discrete and continuous value distributions that are *regular*, ex post efficient allocations are implementable—hence, withholding goods is categorically inefficient—under the maintained assumptions of Bayesian incentive compatibility, ex post budget balance and interim individual rationality. Drexl and Kleiner (2015) establish this result for an allocation problem with two agents and a single unit in which each agent's valuation for the unit has a regular continuous distribution, but their argument generalizes to problems with larger numbers of agents and units. Therefore, the results of Guo and Conitzer (2008, 2009), Moulin (2009) and de Clippel et al. (2014) regarding the necessity of burning money and the optimality of withholding goods in this setting hinge on their imposition of the stronger solution concept of dominant strategy incentive compatibility.

10. Conclusion

In this paper, we investigated the optimality of withholding and damaging resources in the context of Bayesian implementation for bilateral trade and allocation problems with independent private values. In the bilateral trade setting with discrete value distributions, we established that neither burning money nor withholding the good can enhance welfare. We found that the welfare effects of damaging the good are driven by the asymmetry in property rights between the two traders: damage for the buyer cannot improve welfare, while damage for the seller can. Nevertheless, damaging the good for the seller may enhance welfare only if it hurts a seller type with a low valuation more than the type with the highest valuation. We hope that the perturbation techniques we developed in this analysis will prove useful in other mechanism design problems with discrete types.

Extending our results to the bilateral trade setting with continuous value distributions required introducing some hypotheses about virtual values. It is an open question whether these hypotheses are necessary for the continuous-type results.

We also expanded our analysis to allocation problems without property rights. In this environment, we showed that damaging the good for any agent does not increase welfare, mirroring the result from the bilateral trade model that concerns damage for the buyer. We argued that money burning is never optimal, while withholding goods is not optimal if agents have regular value distributions. The divergence of these conclusions from existing research can be attributed to our focus on Bayesian implementation in contrast to the more demanding criterion of dominant strategy implementation (coupled with the desideratum of no ex post budget deficit) commonly imposed in the literature.

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

Appendix. Proofs

Proof of Proposition 1. Let (x_b, x_s, t_b, t_s) be an optimal implementable mechanism. We prove by contradiction that $t_b(b_i, s_j) = t_s(b_i, s_j)$ for all pairs of types (b_i, s_j). Suppose that there exists a pair (b_i, s_j) with $\delta := t_b(b_i, s_j) - t_s(b_i, s_j) > 0$. Consider a perturbation (x_b, x_s, t'_b, t'_s) of the mechanism (x_b, x_s, t_b, t_s) specifying that the transfer to seller type s_j is increased to match the transfer from b_i when the buyer reports type b_i and that the transfers s_j receives from all buyer types are uniformly reduced and credited to the buyer, so that s_j has the same expected utility following the perturbation. The perturbed mechanism (x_b, x_s, t'_b, t'_s) is identical to (x_b, x_s, t_b, t_s) except for the following profiles of reports:

$$t'_s(b_i, s_j) = t_s(b_i, s_j) + \delta - p_b(b_i)\delta$$
$$t'_s(b_k, s_j) = t_s(b_k, s_j) - p_b(b_i)\delta, \forall k = \overline{1, m}, k \neq i$$
$$t'_b(b_k, s_j) = t_b(b_k, s_j) - p_b(b_i)\delta, \forall k = \overline{1, m}.$$

The definition is so that $(\bar{x}_b, \bar{x}_s, \bar{t}'_b, \bar{t}'_s)$ is identical to $(\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s)$ at all values with the following exception:

$$\bar{t}'_b(b_k) = \bar{t}_b(b_k) - p_b(b_i)p_s(s_j)\delta, \forall k = \overline{1, m}.$$

[13] Propositions 4, 5 and 9 do not have natural equivalents in this environment because the seller's ownership of the good is central to our analysis of damage for the seller. All agents act as buyers here.

We can easily compare expected outcomes under the mechanism (x_b, x_s, t_b', t_s') to those under (x_b, x_s, t_b, t_s). Each buyer type receives the good with the same probability and expects a uniform decrease in transfers. Each seller type obtains the good with the same probability and receives the same expected transfer. Therefore, (x_b, x_s, t_b', t_s') satisfies all buyer and seller rationality and incentive constraints because (x_b, x_s, t_b, t_s) does. We conclude that the mechanism (x_b, x_s, t_b', t_s') is implementable and yields an increase in welfare of $p_b(b_i)p_s(s_j)\delta > 0$ over the mechanism (x_b, x_s, t_b, t_s), a contradiction with the optimality of the latter. □

Proof of Lemma 1. We need to show that the set of constraints enumerated in the statement imply all the other ones. $IC_{b_i \to b_{i+1}}$ and $IC_{b_{i+1} \to b_i}$ lead to

$$b_i(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)) \le \bar{t}_b(b_{i+1}) - \bar{t}_b(b_i) \le b_{i+1}(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)), \forall i = \overline{1, m-1}. \tag{5}$$

In particular, $b_i(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)) \le b_{i+1}(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i))$ and $b_{i+1} \ge b_i$ imply that

$$\bar{x}_b(b_{i+1}) \ge \bar{x}_b(b_i), \forall i = \overline{1, m-1}. \tag{6}$$

If $IC_{b_{i+1} \to b_i}$ holds with equality, then the second inequality in (5) holds with equality. If additionally $\bar{x}_b(b_{i+1}) > \bar{x}_b(b_i)$, then the first expression in (5) is strictly smaller than the last expression in (5), so the first inequality must be strict, which means that $IC_{b_i \to b_{i+1}}$ is satisfied with strict inequality.

If $i < k \le m$, then (5) and (6) imply that

$$\begin{aligned} b_i(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)) &\le b_g(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)) \le \bar{t}_b(b_{g+1}) - \bar{t}_b(b_g) \\ &\le b_{g+1}(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)) \le b_k(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)), \forall g = \overline{i, k-1}. \end{aligned}$$

Adding up these inequalities, we obtain

$$b_i(\bar{x}_b(b_k) - \bar{x}_b(b_i)) \le \bar{t}_b(b_k) - \bar{t}_b(b_i) \le b_k(\bar{x}_b(b_k) - \bar{x}_b(b_i)),$$

which is equivalent to $IC_{b_i \to b_k}$ and $IC_{b_k \to b_i}$.

Also, $IC_{b_k \to b_1}$ and IR_{b_1} imply that

$$b_k \bar{x}_b(b_k) - \bar{t}_b(b_k) \ge b_k \bar{x}_b(b_1) - \bar{t}_b(b_1) \ge b_1 \bar{x}_b(b_1) - \bar{t}_b(b_1) \ge 0,$$

verifying IR_{b_k} for $k = \overline{2, m}$.

The claims regarding the seller are checked similarly. □

Proof of Lemma 2. Among the monetary transfer functions t that implement (x_b, x_s), there exists one, t', that maximizes the expected payoff of seller type s_n. We show that t' has the desired properties. Suppose that $IC_{b_{i+1} \to b_i}$ holds with strict inequality under (x_b, x_s, t'). For $\varepsilon > 0$ sufficiently small, the transfer function t'' defined by

$$t''(b_k, s_j) = t'(b_k, s_j) \text{ for } k \le i,\ j = \overline{1, n}$$
$$t''(b_k, s_j) = t'(b_k, s_j) + \varepsilon \text{ for } k \ge i+1,\ j = \overline{1, n}$$

satisfies the sufficient individual rationality and incentive constraints from Lemma 1, and increases the expected payoff of s_n by $\varepsilon \sum_{k=i+1}^{n} p_b(b_k) > 0$, a contradiction with the definition of t'. The conclusion regarding IR_{b_1} follows from perturbing the transfers as in the formulae above with $i = 0$.

We obtain a similar contradiction if (x_b, x_s, t') satisfies $IC_{s_j \to s_{j+1}}$ with strict inequality by perturbing the transfer function as follows:

$$t''(b_i, s_l) = t'(b_i, s_l) - \frac{\varepsilon}{\sum_{l'=1}^{j} p_s(s_{l'})} \text{ for } l \le j,\ i = \overline{1, m}$$
$$t''(b_i, s_l) = t'(b_k, s_j) + \frac{\varepsilon}{\sum_{l'=j+1}^{n} p_s(s_{l'})} \text{ for } l \ge j+1,\ i = \overline{1, m}.$$

This perturbation keeps buyer's incentives in place because the expected transfer of each buyer type under t'' is the same as under t. □

Proof of Proposition 2. We prove the second part of the statement proceeding by contradiction. Suppose that $s_1 > 0$, and let (x_b, x_s, t) be an optimal implementable mechanism under which the good is withheld with positive probability for some buyer-seller value profiles. We can assume without loss of generality that t is selected such that (x_b, x_s, t) satisfies all the constraints in Lemma 2 with equality. Let (b_i, s_j) be a pair of buyer and seller types for which the good is withheld with positive probability, i.e., $x_b(b_i, s_j) + x_s(b_i, s_j) < 1$. Let j' be the highest seller type such that $\bar{x}_s(s_j) = \bar{x}_s(s_{j'})$. By Lemma 1, since (x_b, x_s, t) satisfies $IC_{s_{j'} \to s_{j'+1}}$ with equality and $\bar{x}_s(s_{j'}) < \bar{x}_s(s_{j'+1})$, it must satisfy $IC_{s_{j'+1} \to s_{j'}}$ with strict inequality.

Step 1. We show that $j \neq j'$ and $j \neq n$.

For a proof by contradiction, suppose that $j = j'$ or $j = n$. Consider the mechanism (x'_b, x'_s, t'), which perturbs (x_b, x_s, t) at the following values:

$$x'_s(b_i, s_j) = x_s(b_i, s_j) + \varepsilon$$
$$t'(b_k, s_j) = t(b_k, s_j) - s_j p_b(b_i)\varepsilon, \forall k = \overline{1, m}.$$

For small $\varepsilon > 0$, $x'_b(b_i, s_j) + x'_s(b_i, s_j) \leq 1$. The definition is so that $(\bar{x}'_b, \bar{x}'_s, \bar{t}'_b, \bar{t}'_s)$ take the same values as $(\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s)$ with the following exceptions:

$$\bar{x}'_s(s_j) = \bar{x}_s(s_j) + p_b(b_i)\varepsilon$$
$$\bar{t}'_s(s_j) = \bar{t}_s(s_j) - s_j p_b(b_i)\varepsilon$$
$$\bar{t}'_b(b_k) = \bar{t}_b(b_k) - s_j p_b(b_i) p_s(s_j)\varepsilon, \forall k = \overline{1, m}.$$

We compare expected outcomes under the mechanism (x'_b, x'_s, t') to those under (x_b, x_s, t). For each type report, the buyer receives the good with the same probability, and the expected transfer is uniformly reduced by a constant. Hence, (x'_b, x'_s, t') satisfies all buyer rationality and incentive constraints because (x_b, x_s, t) does.

The mechanism (x'_b, x'_s, t') satisfies all seller constraints that do not involve type s_j because $(\bar{x}'_s, \bar{t}'_s)$ and $(\bar{x}_s, \bar{t}_s)$ coincide for those types. Moreover, seller type s_j expects the same utility from reporting any given type under (x'_b, x'_s, t') and (x_b, x_s, t), so (x'_b, x'_s, t') also satisfies the constraints involving the rationality and incentives of type s_j. We are left to check that (x'_b, x'_s, t') satisfies the constraints $IC_{s_{j-1} \to s_j}$ and $IC_{s_{j+1} \to s_j}$ from Lemma 1.

$IC_{s_{j-1} \to s_j}$ for (x'_b, x'_s, t') is equivalent to

$$\bar{t}_s(s_{j-1}) - (1 - \bar{x}_s(s_{j-1}))s_{j-1} \geq (\bar{t}_s(s_j) - s_j p_b(b_i)\varepsilon) - (1 - (\bar{x}_s(s_j) + p_b(b_i)\varepsilon))s_{j-1}$$
$$\Longleftrightarrow \bar{t}_s(s_{j-1}) - (1 - \bar{x}_s(s_{j-1}))s_{j-1} \geq \bar{t}_s(s_j) - (1 - \bar{x}_s(s_j))s_{j-1} - (s_j - s_{j-1})p_b(b_i)\varepsilon,$$

which follows from $s_j > s_{j-1}$ and the fact that (x_b, x_s, t) satisfies $IC_{s_{j-1} \to s_j}$.

For small $\varepsilon > 0$, (x'_b, x'_s, t') satisfies $IC_{s_{j+1} \to s_j}$ by continuity since (x'_b, x'_s, t') satisfies $IC_{s_{j+1} \to s_j}$ with strict inequality (for $j = j'$, this was argued immediately after the definition of j'; for $j = n$, the argument is unnecessary).

Therefore, for small $\varepsilon > 0$, the mechanism (x'_b, x'_s, t') is implementable and yields an increase in welfare of $s_j p_b(b_i) p_s(s_j)\varepsilon > 0$ over the mechanism (x_b, x_s, t), a contradiction with the optimality of (x_b, x_s, t).

Step 2. It must be that $x_b(b_{i'}, s_{j'}) + x_s(b_{i'}, s_{j'}) = 1$ for all $i' = \overline{1, m}$.

This follows by replacing i with i' and j with j' in the argument from Step 1.

Step 3. We have that $x_b(b_i, s_{j'}) = 0$ and $x_s(b_i, s_{j'}) = 1$.

Steps 1 and 2 show that $j < j'$ and $x_b(b_i, s_{j'}) + x_s(b_i, s_{j'}) = 1$. Suppose, for a contradiction, that $x_b(b_i, s_{j'}) > 0$.

Construct the mechanism (x'_b, x'_s, t') identical to (x_b, x_s, t) on $V_b \times V_s$ with the following exceptions:

$$x'_b(b_i, s_j) = x_b(b_i, s_j) + \varepsilon$$
$$x'_b(b_i, s_{j'}) = x_b(b_i, s_{j'}) - \frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon \quad \& \quad x'_s(b_i, s_{j'}) = x_s(b_i, s_{j'}) + \frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon$$
$$t'(b_k, s_{j'}) = t(b_k, s_{j'}) - s_{j'} p_b(b_i) \frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon, \forall k = \overline{1, m}.$$

For small $\varepsilon > 0$, we have $x'_b(b_i, s_j) + x'_s(b_i, s_j) \leq 1$ and $x'_b(b_i, s_{j'}) \geq 0$.

Arguments similar to those used in Step 1 show that the mechanism (x'_b, x'_s, t') is implementable for small $\varepsilon > 0$. We reach the contradiction that (x'_b, x'_s, t') generates

$$s_{j'} p_s(s_{j'}) p_b(b_i) \frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon = s_{j'} p_b(b_i) p_s(s_j)\varepsilon > 0$$

more surplus than (x_b, x_s, t).

Step 4. We reach the final contradiction.

Steps 1 and 3 establish that $j < j'$ and $x_s(b_i, s_{j'}) = 1$. In particular, $x_s(b_i, s_j) \le x_b(b_i, s_j) + x_s(b_i, s_j) < 1 = x_s(b_i, s_{j'})$. Since $x_s(b_i, s_j) < x_s(b_i, s_{j'})$, $p_b(b_i) > 0$, and $\bar{x}_s(s_j) = \bar{x}_s(s_{j'})$, there exists a buyer type $b_{i'}$ such that $x_s(b_{i'}, s_j) > x_s(b_{i'}, s_{j'})$. It follows that $x_s(b_{i'}, s_j) > 0$, $x_b(b_{i'}, s_j) < 1$, and $x_s(b_{i'}, s_{j'}) < 1$. By Step 2, $x_b(b_{i'}, s_{j'}) + x_s(b_{i'}, s_{j'}) = 1$, and hence $x_b(b_{i'}, s_{j'}) > 0$. We collect the relevant inequalities:

$$x_s(b_i, s_j) < 1,\ x_s(b_{i'}, s_j) > 0,\ x_b(b_{i'}, s_j) < 1,\ x_b(b_{i'}, s_{j'}) > 0,\ x_s(b_{i'}, s_{j'}) < 1. \tag{7}$$

Construct a mechanism (x'_b, x'_s, t') that perturbs (x_b, x_s, t) for the following types:

$$\begin{aligned}
&x'_s(b_i, s_j) = x_s(b_i, s_j) + \varepsilon \\
&x'_s(b_{i'}, s_j) = x_s(b_{i'}, s_j) - \frac{p_b(b_i)}{p_b(b_{i'})}\varepsilon \quad \& \quad x'_b(b_{i'}, s_j) = x_b(b_{i'}, s_j) + \frac{p_b(b_i)}{p_b(b_{i'})}\varepsilon \\
&x'_b(b_{i'}, s_{j'}) = x_b(b_{i'}, s_{j'}) - \frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon \quad \& \quad x'_s(b_{i'}, s_{j'}) = x_s(b_{i'}, s_{j'}) + \frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon \\
&t'(b_k, s_{j'}) = t(b_k, s_{j'}) - s_{j'} p_b(b_{i'})\frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon = t(b_k, s_{j'}) - s_{j'} p_b(b_i)\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon, \forall k = \overline{1, m}.
\end{aligned}$$

For small $\varepsilon > 0$, the set of inequalities (7) guarantees that all values of x' belong to the interval of probabilities $[0, 1]$. We can then argue as in Steps 1 and 3 that for small $\varepsilon > 0$, the mechanism (x'_b, x'_s, t') is implementable and improves the welfare of (x_b, x_s, t) by

$$s_{j'} p_b(b_{i'}) p_s(s_{j'})\frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon = s_{j'} p_s(s_j) p_b(b_i)\varepsilon > 0,$$

a contradiction.

Steps 1 through 4 show that if (x_b, x_s, t) is an optimal implementable mechanism, then $x_b(b_i, s_j) + x_s(b_i, s_j) = 1$.

To establish the first part of the result, note that the set of tuples $(x_b, x_s, \bar{t})$ associated with optimal implementable mechanisms (x_b, x_s, t) is a compact subset of a Euclidean space. It follows that there exists an optimal implementable mechanism that minimizes the total probability of withholding. If this mechanism withholds the good with positive probability, then the arguments above lead to a contradiction (and also imply that withholding under an optimal mechanism is possible only in the case $s_1 = 0$, and withholding may take place in this case only when the seller has type s_1). □

Proof of Proposition 3. As discussed in the proof sketch, it is sufficient to establish the result for cases in which $p_b(b_i) = p'_b(b_i) - \delta$ and $p_b(b_{i+1}) = p'_b(b_{i+1}) + \delta$ with $\delta > 0$, and $p'_b(b_k) = p_b(b_k)$ for $k \ne i, i+1$.

Let (x', t') be a mechanism that maximizes welfare when traders' values are distributed according to (p'_b, p_s) and satisfies all constraints in Lemma 2 with equality (we use only the equality in the constraint $IC_{b_{i+1}\to b_i}$ for this proof). Define the mechanism (x, t) to coincide with (x', t') for all type profiles with the following exceptions:

$$\begin{aligned}
x(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})}x'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}x'(b_{i+1}, s_j), \forall j = \overline{1, n} \\
t(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})}t'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}t'(b_{i+1}, s_j), \forall j = \overline{1, n}.
\end{aligned}$$

Denote by $\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s$ and $\bar{x}'_b, \bar{x}'_s, \bar{t}'_b, \bar{t}'_s$ the probabilities of trade and expected transfers for each buyer and seller type under the mechanism (x, t) when values are distributed according to (p_b, p_s) and under the mechanism (x', t') when values are distributed according to (p'_b, p_s), respectively. The allocation x is defined so that

$$p_b(b_i)x(b_i, s_j) + p_b(b_{i+1})x(b_{i+1}, s_j) = p'_b(b_i)x'(b_i, s_j) + p'_b(b_{i+1})x'(b_{i+1}, s_j),$$

which implies that $\bar{x}_s(s_j) = \bar{x}'_s(s_j)$ for all j. Similarly, $\bar{t}_s(s_j) = \bar{t}'_s(s_j)$ for all j. Moreover, $(\bar{x}'_b, \bar{x}'_s, \bar{t}'_b, \bar{t}'_s)$ is identical to $(\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s)$ at all values except b_{i+1}, for which we have

$$\begin{aligned}
\bar{x}_b(b_{i+1}) &= \frac{\delta}{p_b(b_{i+1})}\bar{x}'_b(b_i) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}\bar{x}'_b(b_{i+1}) \\
\bar{t}_b(b_{i+1}) &= \frac{\delta}{p_b(b_{i+1})}\bar{t}'_b(b_i) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}\bar{t}'_b(b_{i+1}).
\end{aligned}$$

We show that the mechanism (x, t) satisfies all the IR and IC constraints in Lemma 1 when traders' values are distributed according to (p_b, p_s). Since $\bar{x}_b, \bar{t}_b, \bar{x}_s, \bar{t}_s$ coincide with $\bar{x}'_b, \bar{t}'_b, \bar{x}'_s, \bar{t}'_s$ at all values except for b_{i+1}, the only constraints we need to check are those that involve buyer type b_{i+1}. Neither type b_i nor type b_{i+2} have an incentive to mimic type b_{i+1} in the mechanism (x, t) since each of these types is assigned the same allocation probabilities and expected transfers under (x,t) and (x',t'), and prefers his outcome under (x',t') to any other outcome achievable under (x',t') as well as to the convex combination of outcomes for types b_i and b_{i+1} specified by $(x(b_{i+1},\cdot),t(b_{i+1},\cdot))$.

We are left to check that (x,t) satisfies $IC_{b_{i+1}\to b_i}$ and $IC_{b_{i+1}\to b_{i+2}}$. The constraint $IC_{b_{i+1}\to b_i}$ requires that

$$\frac{\delta}{p_b(b_{i+1})}(\bar{x}'_b(b_i)b_{i+1}-\bar{t}'_b(b_i))+\frac{p_b(b_{i+1})-\delta}{p_b(b_{i+1})}(\bar{x}'_b(b_{i+1})b_{i+1}-\bar{t}'_b(b_{i+1}))\geq \bar{x}'_b(b_i)b_{i+1}-\bar{t}'_b(b_i),$$

which is equivalent to

$$x'_b(b_{i+1})b_{i+1}-\bar{t}'_b(b_{i+1})\geq \bar{x}'_b(b_i)b_{i+1}-\bar{t}'_b(b_i).$$

The latter inequality is a consequence of $IC_{b_{i+1}\to b_i}$ under the mechanism (x',t').

To check that (x,t) satisfies $IC_{b_{i+1}\to b_{i+2}}$, we need to show that

$$\frac{\delta}{p_b(b_{i+1})}(\bar{x}'_b(b_i)b_{i+1}-\bar{t}'_b(b_i))+\frac{p_b(b_{i+1})-\delta}{p_b(b_{i+1})}(\bar{x}'_b(b_{i+1})b_{i+1}-\bar{t}'_b(b_{i+1}))\geq \bar{x}'_b(b_{i+2})b_{i+1}-\bar{t}'_b(b_{i+2}).$$

As (x',t') is assumed to satisfy $IC_{b_{i+1}\to b_i}$ with equality, we have $\bar{x}'_b(b_i)b_{i+1}-\bar{t}'_b(b_i)=\bar{x}'_b(b_{i+1})b_{i+1}-\bar{t}'_b(b_{i+1})$, so the inequality above follows from $IC_{b_{i+1}\to b_{i+2}}$ under (x',t').

Since $p_b,p_s,\bar{x}_b,\bar{t}_b,\bar{x}_s,\bar{t}_s$ coincide with $p'_b,p_s,\bar{x}'_b,\bar{t}'_b,\bar{x}'_s,\bar{t}'_s$ at all values except for b_i and b_{i+1}, the difference in the welfare achieved by (x,t) with distributions (p_b,p_s) and (x',t') with distributions (p'_b,p_s) is given by

$$\begin{aligned}
&p_b(b_i)\bar{x}_b(b_i)b_i+p_b(b_{i+1})\bar{x}_b(b_{i+1})b_{i+1}-p'_b(b_i)\bar{x}'_b(b_i)b_i-p'_b(b_{i+1})\bar{x}'_b(b_{i+1})b_{i+1}\\
&=p_b(b_i)\bar{x}'_b(b_i)b_i+p_b(b_{i+1})\left(\frac{\delta}{p_b(b_{i+1})}\bar{x}'_b(b_i)+\frac{p_b(b_{i+1})-\delta}{p_b(b_{i+1})}\bar{x}'_b(b_{i+1})\right)b_{i+1}\\
&-(p_b(b_i)+\delta)\bar{x}'_b(b_i)b_i-(p_b(b_{i+1})-\delta)\bar{x}'_b(b_{i+1})b_{i+1}\\
&=\delta\bar{x}'_b(b_i)(b_{i+1}-b_i),
\end{aligned}$$

which is non-negative. Therefore, the optimal mechanism for (p_b,p_s) yields at least the same amount of welfare as the optimal mechanism for (p'_b,p_s). □

Proof of Proposition 4. Lemma 2 implies the existence of a transfer function t that implements the allocation x for the original pair of value distributions (p_b,p_s) such that (x,t) satisfies $IC_{b_{i+1}\to b_i}$ with equality. Fix $\varepsilon>0$ and let $s'_l=s_l-\varepsilon$ for $l=\overline{1,j}$. Define

$$\varepsilon'=\varepsilon\frac{(\bar{x}_s(s_j)-\bar{x}_s(s_{j+1}))\sum_{l'=1}^{j}p_s(s_{l'})}{p_s(s_1)(b_{i+1}-b_i)\sum_{k'=i+1}^{m}p_b(b_{k'})}. \tag{8}$$

We seek to implement an allocation x' when the values of seller types $l=\overline{1,j}$ are reduced from s_l to s'_l that differs from x (for corresponding seller types) only in that

$$x'(b_i,s'_1)=x(b_i,s_1)+\varepsilon'.$$

For this purpose, we perturb the transfer function t for the following pairs of types:

$$\begin{aligned}
t'(b_i,s'_1)&=t(b_i,s_1)+\varepsilon' b_i\\
t'(b_k,s'_1)&=t(b_k,s_1)-\varepsilon'(b_{i+1}-b_i)\frac{p_s(s_1)}{\sum_{l'=1}^{j}p_s(s_{l'})}-\varepsilon'(b_i-s'_1)\frac{p_b(b_i)}{\sum_{k'=i+1}^{m}p_b(b_{k'})},\ \forall k=\overline{i+1,m}\\
t'(b_k,s'_l)&=t(b_k,s_l)-\varepsilon'(b_{i+1}-b_i)\frac{p_s(s_1)}{\sum_{l'=1}^{j}p_s(s_{l'})},\ \forall k=\overline{i+1,m}, l=\overline{2,j}.
\end{aligned}$$

Transfers are specified so that buyer type b_i pays exactly his gain from the additional probability of receiving the good from seller type s_1. Similarly, we have that $\bar{t}'_s(s'_1)-\bar{t}'_s(s'_2)=\bar{t}_s(s_1)-\bar{t}_s(s_2)+\varepsilon' p_b(b_i)s'_1$, so that the new seller type s'_1 is compensated relative to type s'_2 for the utility lost by trading with buyer type b_i with the extra ε' probability.

To show that (x',t') is implementable following the value reduction, we verify that (x',t') satisfies the sufficient individual rationality and incentive constraints from Lemma 1 as well as the stricter participation constraints required for damaging the good for the seller.

Buyer expectations $(\bar{x}'_b, \bar{t}'_b)$ under the new mechanism differ from the original ones in the following cases:

$$\bar{x}'_b(b_i) = \bar{x}_b(b_i) + \varepsilon' p_s(s_1)$$
$$\bar{t}'_b(b_i) = \bar{t}_b(b_i) + \varepsilon' p_s(s_1) b_i$$
$$\bar{t}'_b(b_k) = \bar{t}_b(b_k) - \varepsilon' p_s(s_1)(b_{i+1} - b_i) - \varepsilon'(b_i - s'_1)\frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})}, \forall k = \overline{i+1, m}.$$

The mechanism (x', t') satisfies the incentive constraints $IC_{b_k \to b_{k+1}}$ and $IC_{b_{k+1} \to b_k}$ for $k = \overline{i+1, m-1}$ since the involved buyer types receive the same allocation under x and x' and their expected payments are reduced by the same amount when shifting from t to t'. It is clear that (x', t') satisfies the constraints $IC_{b_k \to b_{k+1}}$ and $IC_{b_{k+1} \to b_k}$ for $k = \overline{1, i-2}$ and IR_{b_1}.

We next check that (x', t') satisfies the incentive constraints $IC_{b_{i-1} \to b_i}$, $IC_{b_i \to b_{i-1}}$, $IC_{b_i \to b_{i+1}}$ and $IC_{b_{i+1} \to b_i}$. The mechanism (x', t') satisfies $IC_{b_i \to b_{i-1}}$ because buyer type b_i expects the same utility under (x, t) and (x', t') when he reports type b_i, and the same is true when he reports type b_{i-1}. $IC_{b_{i-1} \to b_i}$ holds under (x', t') since for buyer type b_{i-1}, the marginal loss from misreporting his type to be b_i when we shift from (x, t) to (x', t') entails receiving the good with additional probability $\varepsilon' p_s(s_1)$ for an expected benefit of $\varepsilon' p_s(s_1) b_{i-1}$ at the greater cost of $\varepsilon' p_s(s_1) b_i$.

The buyer's incentives we are left to check for (x', t') are $IC_{b_{i+1} \to b_i}$ and $IC_{b_i \to b_{i+1}}$. Since $IC_{b_{i+1} \to b_i}$ holds with equality under (x, t), we have that

$$\bar{x}_b(b_{i+1}) b_{i+1} - \bar{t}_b(b_{i+1}) = \bar{x}_b(b_i) b_{i+1} - \bar{t}_b(b_i). \tag{9}$$

To see that (x', t') satisfies $IC_{b_{i+1} \to b_i}$, we use (9) to infer that

$$\begin{aligned}\bar{x}'_b(b_{i+1})b_{i+1} - \bar{t}'_b(b_{i+1}) &= \bar{x}_b(b_{i+1})b_{i+1} - \bar{t}_b(b_{i+1}) + \varepsilon' p_s(s_1)(b_{i+1} - b_i) + \varepsilon'(b_i - s'_1)\frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} \\ &= \bar{x}_b(b_i)b_{i+1} - \bar{t}_b(b_i) + \varepsilon' p_s(s_1)(b_{i+1} - b_i) + \varepsilon'(b_i - s'_1)\frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} \\ &= \bar{x}'_b(b_i)b_{i+1} - \bar{t}'_b(b_i) + \varepsilon'(b_i - s'_1)\frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} > \bar{x}'_b(b_i)b_{i+1} - \bar{t}'_b(b_i).\end{aligned}$$

The inequality follows from the hypothesis that $b_i > s_1$, which implies that $b_i > s'_1$.

Also using (9), we obtain that

$$\begin{aligned}&\bar{x}'_b(b_i)b_i - \bar{t}'_b(b_i) = \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) = \bar{x}_b(b_{i+1})b_i - \bar{t}_b(b_{i+1}) + (b_{i+1} - b_i)(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)) \\ &= \bar{x}'_b(b_{i+1})b_i - \bar{t}'_b(b_{i+1}) - \varepsilon' p_s(s_1)(b_{i+1} - b_i) - \varepsilon'(b_i - s'_1)\frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} + (b_{i+1} - b_i)(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)).\end{aligned}$$

$IC_{b_i \to b_{i+1}}$ for (x', t') is thus equivalent to

$$\varepsilon' p_s(s_1)(b_{i+1} - b_i) + \varepsilon'(b_i - s'_1)\frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} \le (b_{i+1} - b_i)(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)),$$

which holds for small $\varepsilon > 0$ when ε' is given by (8) since $\bar{x}_b(b_{i+1}) > \bar{x}_b(b_i)$ implies that the right-hand side of the inequality is positive.

We now turn to verifying the seller constraints for (x', t'). Seller expectations $(\bar{x}'_s, \bar{t}'_s)$ under the perturbed mechanism differ from the corresponding ones in the original mechanism in the following instances:

$$\bar{x}'_s(s'_1) = \bar{x}_s(s_1) + \varepsilon' p_b(b_i)$$
$$\bar{t}'_s(s'_1) = \bar{t}_s(s_1) + \varepsilon' p_b(b_i) s'_1 - \varepsilon' p_s(s_1)(b_{i+1} - b_i)\frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})}$$
$$\bar{t}'_s(s'_l) = \bar{t}_s(s_l) - \varepsilon' p_s(s_1)(b_{i+1} - b_i)\frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})}, \forall l = \overline{2, j}.$$

We provide a proof for $j \geq 2$ (the case $j = 1$ involves similar arguments). Since $\bar{x}_s(s_1) = \bar{x}_s(s_j)$, it must be that $\bar{x}_s(s_l) = \bar{x}_s(s_1)$ and $\bar{t}_s(s_l) = \bar{t}_s(s_1)$ for $l = \overline{2, j}$. Hence $\bar{x}'_s(s_l) = \bar{x}'_s(s_2)$ and $\bar{t}'_s(s_l) = \bar{t}'_s(s_2)$ for $l = \overline{2, j}$. This implies that (x', t') satisfies $IC_{s'_l \to s'_{l+1}}$ and $IC_{s'_{l+1} \to s'_l}$ for $l = \overline{2, j-1}$. As $(\bar{x}'_s, \bar{t}'_s)$ coincides with $(\bar{x}_s, \bar{t}_s)$ for seller types $l = \overline{j+1, n}$, and (x, t) is implementable, (x', t') satisfies $IC_{s_l \to s_{l+1}}$ and $IC_{s_{l+1} \to s_l}$ for $l = \overline{j+1, n-1}$ and IR_{s_n}. We are left to check the following incentive compatibility constraints from Lemma 1: $IC_{s'_1 \to s'_2}$, $IC_{s'_2 \to s'_1}$, $IC_{s'_j \to s_{j+1}}$ and $IC_{s_{j+1} \to s'_j}$.

The constraints $IC_{s'_1 \to s'_2}$ and $IC_{s'_2 \to s'_1}$ for (x', t') are equivalent to

$$(\bar{x}'_s(s'_1) - \bar{x}'_s(s'_2))s'_1 \le \bar{t}'_s(s'_1) - \bar{t}'_s(s'_2) \le (\bar{x}'_s(s'_1) - \bar{x}'_s(s'_2))s'_2.$$

These inequalities follow from $\bar{x}'_s(s'_1) - \bar{x}'_s(s'_2) = \varepsilon' p_b(b_i)$, $\bar{t}'_s(s'_1) - \bar{t}'_s(s'_2) = \varepsilon' p_b(b_i)s'_1$ and $s'_1 < s'_2$.

To check $IC_{s'_j \to s_{j+1}}$ for the mechanism (x', t'), note that $IC_{s_j \to s_{j+1}}$ for (x, t) implies that

$$\begin{aligned}
&\bar{t}'_s(s'_j) - \bar{t}'_s(s'_{j+1}) - (\bar{x}'_s(s'_j) - \bar{x}'_s(s'_{j+1}))s'_j \\
&\quad = \bar{t}_s(s_j) - \bar{t}_s(s_{j+1}) - (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_j + \varepsilon(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) - \varepsilon' p_s(s_1)(b_{i+1} - b_i)\frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})} \\
&\quad \ge \varepsilon(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) - \varepsilon' p_s(s_1)(b_{i+1} - b_i)\frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})} = 0,
\end{aligned}$$

where the last equality follows from the definition of ε' in (8). The slack created in $IC_{s_j \to s_{j+1}}$ by the ε value reduction for seller type j is used here to decrease the transfers from buyer types $i+1$ through m to seller types 1 through j, which makes it possible to introduce the slack in $IC_{b_{i+1} \to b_i}$ necessary for increasing the probability of trade between seller type s_1 and buyer type b_i at terms that would be otherwise attractive to buyer type b_{i+1}.

The mechanism (x', t') satisfies $IC_{s_{j+1} \to s'_j}$ because (x, t) satisfies $IC_{s_{j+1} \to s_j}$ and we have that $\bar{x}'_s(s_{j+1}) = \bar{x}_s(s_{j+1})$, $\bar{t}'_s(s_{j+1}) = \bar{t}_s(s_{j+1})$, $\bar{x}'_s(s'_j) = \bar{x}_s(s_j)$ and $\bar{t}'_s(s'_j) < \bar{t}_s(s_j)$.

The arguments above establish that (x', t') is implementable for the pair of value distributions (p'_b, p_s). We finally check that seller types $l = \overline{1, j}$ have incentives to participate in the damage of the good. Since (x', t') satisfies $IC_{s'_l \to s_{j+1}}$ and $IR_{s_{j+1}}$ and $\bar{x}'_s(s_{j+1}) = \bar{x}_s(s_{j+1}) > 0$, we have that

$$\begin{aligned}
\bar{t}'_s(s'_l) - \bar{x}'_s(s'_l)s'_l &\ge \bar{t}'_s(s_{j+1}) - \bar{x}'_s(s_{j+1})s'_l = \bar{t}'_s(s_{j+1}) - \bar{x}'_s(s_{j+1})s_{j+1} + \bar{x}_s(s_{j+1})(s_{j+1} - s'_l) \\
&\ge \bar{x}_s(s_{j+1})(s_{j+1} - s'_l) > \bar{x}_s(s_{j+1})(s_{j+1} - s_j).
\end{aligned}$$

Then,

$$\bar{t}'_s(s'_l) + (1 - \bar{x}_s(s'_l))s'_l - s_l = \bar{t}'_s(s'_l) - \bar{x}_s(s'_l)s'_l + s'_l - s_l > \bar{x}_s(s_{j+1})(s_{j+1} - s_j) - \varepsilon.$$

The last term is positive if $\varepsilon < \bar{x}_s(s_{j+1})(s_{j+1} - s_j)$, and the proof is completed by noting that $\bar{x}_s(s_{j+1})(s_{j+1} - s_j) > 0$. □

Proof for footnote 4. Suppose that $x(b_i, s_1) = 1$ and $\bar{x}_s(s_1) < 1$. Then, there exists a buyer type i' such that $x(b_{i'}, s_1) < 1$. Since $\bar{x}_b(b_i) = \bar{x}_b(b_1) \le \bar{x}_b(b_{i'})$ and $x(b_{i'}, s_1) < 1 = x(b_i, s_1)$, there must exist a seller type j' such that $x(b_{i'}, s_{j'}) > x(b_i, s_{j'})$. In particular, we have $x(b_{i'}, s_{j'}) > 0$ and $x(b_i, s_{j'}) < 1$. Construct a perturbation x^1 of the allocation x that differs from x only in the following cases:

$$\begin{aligned}
x^1(b_i, s_1) &= x(b_i, s_1) - \eta \\
x^1(b_i, s_{j'}) &= x(b_i, s_{j'}) + \frac{p_s(s_1)}{p_s(s_{j'})}\eta \\
x^1(b_{i'}, s_{j'}) &= x(b_{i'}, s_{j'}) - \frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_1)}{p_s(s_{j'})}\eta \\
x^1(b_{i'}, s_1) &= x(b_{i'}, s_1) + \frac{p_b(b_i)}{p_b(b_{i'})}\eta.
\end{aligned}$$

For sufficiently small $\eta > 0$, the inequalities above imply that all the values of x^1 are in $[0, 1]$. Moreover, x^1 generates the same allocation probability as x for each buyer and seller type, and has the property that $x^1(b_i, s_1) < 1$. The conclusion of footnote 4 then follows from applying Proposition 4 to allocation x^1. □

Computations for Proposition 4 and footnote 6. We compute the optimal implementable mechanisms in the example used to demonstrate Proposition 4 and in an example supporting footnote 6. Assume that the buyer's valuations are $b_1 = 1$ and $b_2 = 4$ with probability 1/2 each, and that the seller's valuations are $s_1 = \varepsilon_1$ and $s_2 = 3 + \varepsilon_2$ with probability 1/2 each, where $\varepsilon_1, \varepsilon_2$ are parameters in $[0, 1)$. Consider an implementable mechanism (x, t) that satisfies the constraints from Lemma 2 with equality. The total welfare created by the mechanism is

$$\frac{1}{4}\left((1 - \varepsilon_1)x(b_1, s_1) - (2 + \varepsilon_2)x(b_1, s_2) + (4 - \varepsilon_1)x(b_2, s_1) + (1 - \varepsilon_2)x(b_2, s_2) + 2(\varepsilon_1 + 3 + \varepsilon_2)\right). \quad (10)$$

Since the mechanism satisfies the constraints from Lemma 2 with equality as well as IR_{s_2}, the following conditions hold:

$$t(b_1,s_1)+t(b_1,s_2)=x(b_1,s_1)+x(b_1,s_2)$$
$$t(b_1,s_1)+t(b_1,s_2)-t(b_2,s_1)-t(b_2,s_2)=4(x(b_1,s_1)+x(b_1,s_2)-x(b_2,s_1)-x(b_2,s_2))$$
$$t(b_1,s_2)+t(b_2,s_2)\geq(3+\varepsilon_2)(x(b_1,s_2)+x(b_2,s_2))$$
$$t(b_1,s_1)-t(b_1,s_2)+t(b_2,s_1)-t(b_2,s_2)=\varepsilon_1(x(b_1,s_1)-x(b_1,s_2)+x(b_2,s_1)-x(b_2,s_2)).$$

These constraints correspond in order to $IR_{b_1}, IC_{b_2\to b_1}, IR_{s_2}$ and $IC_{s_1\to s_2}$ (multiplied by 2). Multiplying the constraints by $-2, 1, 2, 1$, respectively, and adding them up we obtain

$$(4-\varepsilon_1)x(b_2,s_1)\geq(2+\varepsilon_1)x(b_1,s_1)+(8+2\varepsilon_2-\varepsilon_1)x(b_1,s_2)+(2+2\varepsilon_2-\varepsilon_1)x(b_2,s_2). \tag{11}$$

This is the aggregate constraint discussed in Section 8 corresponding to this example.

Given constraint (11), the objective function (10) is maximized only if $x(b_2,s_1)=1$ and $x(b_1,s_2)=0$. Then, (11) reduces to

$$4-\varepsilon_1\geq(2+\varepsilon_1)x(b_1,s_1)+(2+2\varepsilon_2-\varepsilon_1)x(b_2,s_2). \tag{12}$$

Therefore, there is a linear trade-off between $x(b_1,s_1)$ and $x(b_2,s_2)$ when maximizing (10) with relative weights $(1-\varepsilon_1)/(2+\varepsilon_1)$ and $(1-\varepsilon_2)/(2+2\varepsilon_2-\varepsilon_1)$. If $\varepsilon_1>\varepsilon_2$, then the latter weight dominates the former, and setting $x(b_2,s_2)=1$ is optimal. If $\varepsilon_1<\varepsilon_2$, then the former weight dominates, and $x(b_1,s_1)=1$ is optimal. If $\varepsilon_1=\varepsilon_2$, then the two weights are equal, and any specification of $x(b_1,s_1)$ and $x(b_2,s_2)$ such that $x(b_1,s_1)+x(b_2,s_2)=(4-\varepsilon_1)/(2+\varepsilon_1)$ is optimal.

We now specialize the analysis to two cases. In the example illustrating Proposition 4, we have that $\varepsilon_1\geq\varepsilon_2=0$. In this case, the allocation x maximizing (10) subject to (12) is given by

$$x(b_1,s_1)=\frac{2}{2+\varepsilon_1},\ x(b_1,s_2)=0,\ x(b_2,s_1)=1,\ x(b_2,s_2)=1.$$

It can be easily checked that this allocation can be implemented with transfers t for which the constraints $IR_{b_1}, IC_{b_2\to b_1}, IR_{s_2}$ and $IC_{s_1\to s_2}$ are binding. The optimal mechanism generates welfare $(24+11\varepsilon_1+\varepsilon_1^2)/(4(2+\varepsilon_1))$.

For an example corroborating footnote 6, assume that $\varepsilon_1\leq\varepsilon_2$. Then, there exists an optimal mechanism that implements the allocation x given by

$$x(b_1,s_1)=1,\ x(b_1,s_2)=0,\ x(b_2,s_1)=1,\ x(b_2,s_2)=\frac{2-2\varepsilon_1}{2+2\varepsilon_2-\varepsilon_1}.$$

If $\varepsilon_1=\varepsilon_2$, then this allocation achieves a total welfare of $(24+11\varepsilon_1+4\varepsilon_1^2)/(4(2+\varepsilon_1))$, which is decreasing for small $\varepsilon_1\geq 0$. For instance, the welfare generated under the optimal x is 3 for $\varepsilon_1=0$ and approximately 2.993 for $\varepsilon_1=0.1$. Following the damage for the seller implied by the decrease in the common value of ε_1 and ε_2 from 0.1 to 0, the optimal mechanism satisfies the seller's ex ante damage participation constraint described in footnote 6. Indeed, the seller's expected payoff from participating in the post-damage optimal mechanism is 2.25, which is greater than her expected value 1.6 for the undamaged good. □

Proof of Proposition 5. Fix a buyer value distribution p_b, and consider a seller value distribution p'_s with support $s'_1<s'_2<\ldots<s'_n$ that reflects damage at the top relative to p_s with support $s_1<s_2<\ldots<s_n$. Let (x',t'') be an implementable mechanism for the pair of distributions (p_b,p'_s) that satisfies the seller's damage participation constraint for the value reduction from p_s to p'_s. As in the proof of Lemma 2, any implementable mechanism for the pair of distributions (p_b,p'_s) that implements x' (not necessarily satisfying the seller's damage participation constraints) and maximizes the expected payoff of seller type s'_n, satisfies all constraints $IC_{s_j\to s_{j+1}}$ with equality. Let (x',t') be such a mechanism. By definition, (x',t') generates at least the same utility as (x',t'') for type s'_n. Since (x',t'') obeys the seller's damage participation constraint, it delivers utility of at least s_n to the seller of type s'_n, so the same must be true of (x',t'). It follows that $(1-\bar{x}'_s(s'_n))s'_n+\bar{t}'_s(s'_n)\geq s_n$.

We now construct a mechanism (x,t) for the pair of distributions (p_b,p_s) that implements the same type-by-type allocation x', i.e., $x(b_i,s_j)=x'(b_i,s'_j)$ for all i and j, and perturbs the transfer function t' as follows:

$$t(b_i,s_j)=t'(b_i,s'_j)-(1-\bar{x}_s(s_n))(s_n-s'_n)+\sum_{l=j}^{n-1}(\bar{x}_s(s_l)-\bar{x}_s(s_{l+1}))(s_l-s'_l),\ \forall i=\overline{1,m},\ j=\overline{1,n}.$$

Since x and x' represent the same allocation of the good for any profile of corresponding types, and seller values under p_s are higher than under p'_s, the mechanism (x',t') with value distributions (p_b,p'_s) does not achieve greater welfare than the mechanism (x,t) with value distributions (p_b,p_s).

We next argue that (x,t) is an implementable mechanism for the pair of distributions (p_b, p_s) by verifying that it satisfies the constraints from Lemma 1. The mechanism (x,t) satisfies the incentive compatibility constraints for the buyer because the corresponding constraints hold under (x',t'), and $\bar{x}_b$ coincides with $\bar{x}'_b$, while $\bar{t}_b(b_i) - \bar{t}'_b(b_i) = \bar{t}_b(b_k) - \bar{t}'_b(b_k)$ for $i,k = \overline{1,m}$. The mechanism (x,t) satisfies IR_{b_1} because (x',t') does, and $t(b_1,s_j) \le t'(b_1,s_j)$ for $j = \overline{1,n}$. The latter inequality holds because

$$\sum_{l=j}^{n-1}(\bar{x}_s(s_l) - \bar{x}_s(s_{l+1}))(s_l - s'_l) \le \sum_{l=j}^{n-1}(\bar{x}_s(s_l) - \bar{x}_s(s_{l+1}))(s_n - s'_n)$$
$$= (\bar{x}_s(s_j) - \bar{x}_s(s_n))(s_n - s'_n) \le (1 - \bar{x}_s(s_n))(s_n - s'_n), \forall j = \overline{1,n}$$

as $s_l - s'_l \le s_n - s'_n$, $\bar{x}_s(s_l) - \bar{x}_s(s_{l+1}) \ge 0$ for $l = \overline{j, n-1}$, and $\bar{x}_s(s_j) \le 1$.

We are left to check that (x,t) satisfies the constraints IR_{s_n}, $IC_{s_j \to s_{j+1}}$, and $IC_{s_{j+1} \to s_j}$ for all j. The constraint IR_{s_n} holds under (x,t) since the utility of type s_n under (x,t) is

$$(1 - \bar{x}_s(s_n))s_n + \bar{t}(s_n) = (1 - \bar{x}'_s(s'_n))s_n + \bar{t}'_s(s'_n) - (1 - \bar{x}_s(s_n))(s_n - s'_n) = (1 - \bar{x}'_s(s'_n))s'_n + \bar{t}'_s(s'_n) \ge s_n,$$

where the inequality was derived above.

To show that (x,t) satisfies $IC_{s_j \to s_{j+1}}$ and $IC_{s_{j+1} \to s_j}$, note that the fact that $IC_{s_j \to s_{j+1}}$ is binding under (x',t') means that

$$\bar{t}'_s(s'_j) - \bar{t}'_s(s'_{j+1}) = (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s'_j.$$

It follows that

$$\bar{t}_s(s_j) - \bar{t}_s(s_{j+1}) = \bar{t}'_s(s'_j) - \bar{t}'_s(s'_{j+1}) + (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))(s_j - s'_j) = (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_j.$$

Hence,

$$(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_j = \bar{t}_s(s_j) - \bar{t}_s(s_{j+1}) \le (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_{j+1}.$$

This proves that (x,t) satisfies $IC_{s_j \to s_{j+1}}$ and $IC_{s_{j+1} \to s_j}$.

We have argued that (x,t) is an implementable mechanism for the pair of value distributions (p_b, p_s) that generates at least the same amount of welfare as (x',t') for the pair of value distributions (p_b, p'_s). Therefore, the optimal implementable mechanism for the value distributions (p_b, p_s) achieves at least the same amount of welfare as the mechanism (x',t') for the value distributions (p_b, p'_s). □

Proof of Proposition 6. Let (x_b, x_s, t_b, t_s) be an optimal implementable mechanism. Suppose, for a contradiction, that

$$\Delta := \int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(b,s) - t_s(b,s)) f_b(b) f_s(s)\, db\, ds > 0.$$

Perturb transfers as follows:

$$t'_b(b,s) = t'_s(b,s) = t_b(b,s) - \int_{\underline{b}}^{\bar{b}} (t_b(\tilde{b},s) - t_s(\tilde{b},s)) f_b(\tilde{b})\, d\tilde{b}.$$

Expected transfers under this perturbation are given by

$$\bar{t}'_b(b) = \bar{t}_b(b) - \int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(\tilde{b},s) - t_s(\tilde{b},s)) f_b(\tilde{b}) f_s(s)\, d\tilde{b}\, ds, \forall b \in [\underline{b}, \bar{b}]$$
$$\bar{t}'_s(s) = \bar{t}_s(s), \forall s \in [\underline{s}, \bar{s}].$$

Hence, under the perturbation, expected transfers to each seller type are the same as under the original mechanism, while expected transfers from each buyer type are uniformly decreased by $\int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(\tilde{b},s) - t_s(\tilde{b},s)) f_b(\tilde{b}) f_s(s)\, d\tilde{b}\, ds = \Delta$. Therefore, the mechanism (x_b, x_s, t'_b, t'_s) is implementable and yields an increase in welfare of $\Delta > 0$ over the mechanism (x_b, x_s, t_b, t_s), a contradiction with the optimality of the latter. □

Proof of Proposition 7. Let (x_b, x_s) be an implementable allocation for the pair of regular value distributions (F_b, F_s). Let $U_b(b)$ and $U_s(s)$ denote the utility the buyer of type b and the seller of type s gain from participating in the mechanism (in addition to s for the seller), respectively. As in Myerson (1981) and Myerson and Satterthwaite (1983), revenue equivalence leads to

$$U_b(b) = U_b(\underline{b}) + \int_{\underline{b}}^{b} \bar{x}_b(b) f_b(b)\, db$$

$$U_s(s) = U_s(\bar{s}) + \int_{s}^{\bar{s}} (1 - \bar{x}_s(s)) f_s(s)\, ds.$$

Then, the gains from trade $\int_{\underline{b}}^{\bar{b}} U_b(b) f_b(b)\, db + \int_{\underline{s}}^{\bar{s}} U_s(s) f_s(s)\, ds$ achieved by a mechanism implementing (x_b, x_s) are given by

$$U_b(\underline{b}) + U_s(\bar{s}) + \int_{\underline{b}}^{\bar{b}} (1 - F_b(b))\bar{x}_b(b) f_b(b)\, db + \int_{\underline{s}}^{\bar{s}} F_s(s)(1 - \bar{x}_s(s)) f_s(s)\, ds.$$

Since the gains from trade can be computed also as

$$\int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\bar{s}} (x_b(b,s)b - (1 - x_s(b,s))s)\, f_b(b) f_s(s)\, ds\, db, \tag{13}$$

we get that

$$U_b(\underline{b}) + U_s(\bar{s}) = \int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\bar{s}} \left(x_b(b,s) \left(b - \frac{1 - F_b(b)}{f_b(b)} \right) - (1 - x_s(b,s)) \left(s + \frac{F_s(s)}{f_s(s)} \right) \right) f_b(b) f_s(s)\, ds\, db \geq 0. \tag{14}$$

The proof of Theorem 1 in Myerson and Satterthwaite (1983) can be easily adapted to show that an allocation (x_b, x_s) is implementable if and only if inequality (14) holds, $x_b(b,s) + x_s(b,s) \leq 1$ for all pairs of types (b,s), and $\bar{x}_b$ and $\bar{x}_s$ are increasing functions (recall that in our setting $\bar{x}_s(s)$ represents the probability that seller type s keeps the good, and not the probability that type s trades the good, which is the primitive variable in the analysis of Myerson and Satterthwaite; the two variables have opposite monotonicity).

Consider now the relaxed optimization problem of maximizing the gains from trade given by formula (13)—which is equivalent to maximizing total welfare—subject to (14) and $x_b(b,s) + x_s(b,s) \leq 1$ for all (b,s). If $\lambda \geq 0$ denotes the Lagrange multiplier on constraint (14), we obtain the Lagrangian

$$(\lambda + 1) \int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\bar{s}} \left(x_b(b,s) \left(b - \alpha \frac{1 - F_b(b)}{f_b(b)} \right) - (1 - x_s(b,s)) \left(s + \alpha \frac{F_s(s)}{f_s(s)} \right) \right) f_b(b) f_s(s)\, ds\, db,$$

where $\alpha := \lambda/(\lambda + 1) \in [0, 1)$.

Let $c_b(b, \alpha)$ and $c_s(s, \alpha)$ denote the coefficients multiplying the terms $x_b(b,s)$ and $-(1 - x_s(b,s))$ in the integrand above, respectively. Choosing $x_b(b,s)$ and $x_s(b,s)$ such that $x_b(b,s) + x_s(b,s) \leq 1$ to maximize the integrand pointwise for every pair (b,s) requires that if $c_b(b,\alpha) > c_s(s,\alpha)$, then $x_b(b,s) = 1$ and $x_s(b,s) = 0$, and if $c_b(b,\alpha) < c_s(s,\alpha)$, then $x_b(b,s) = 0$ and $x_s(b,s) = 1$. Let x^α be an allocation with these properties.

As in the argument for Theorem 2 of Myerson and Satterthwaite, the assumption that F_b and F_s are regular distributions implies that for any $\alpha \in [0,1)$, $c_b(\cdot, \alpha)$ and $c_s(\cdot, \alpha)$ are strictly increasing functions, and hence $\bar{x}_b^\alpha$ and $\bar{x}_s^\alpha$ must be increasing. Moreover, when $[\underline{b}, \bar{b}] \cap [\underline{s}, \bar{s}] \neq \emptyset$,[14] there exists a unique $\alpha^* \in [0,1)$ for which constraint (14) holds with equality under $(x_b^{\alpha^*}, x_s^{\alpha^*})$. It follows that $(x_b^{\alpha^*}, x_s^{\alpha^*})$ solves the relaxed optimization problem, and given the monotonicity of $\bar{x}_b^{\alpha^*}$ and $\bar{x}_s^{\alpha^*}$, is an optimal implementable allocation. Moreover, any optimal mechanism must implement an allocation that coincides with $(x_b^{\alpha^*}, x_s^{\alpha^*})$ with probability 1 under $F_b \times F_s$.

The proof is completed by noting that $x_b^{\alpha^*}(b,s) + x_s^{\alpha^*}(b,s) = 1$ for all pairs (b,s) such that $c_b(b,\alpha^*) \neq c_s(s,\alpha^*)$, and that the latter condition holds with probability 1 (since $c_b(\cdot,\alpha)$ and $c_s(\cdot,\alpha)$ are strictly increasing functions, for every b there is at most one s such that $c_b(b,\alpha^*) = c_s(s,\alpha^*)$, and vice versa). □

[14] The case of non-overlapping supports is trivial.

Proof of Proposition 8. Let B and S be random variables with distributions F_b and F_s, respectively. Then, $B' := \alpha(B)$ is a random variable distributed according to G_b. Since F_b first-order stochastically dominates G_b, and $\alpha = G_b^{-1} \circ F_b$, it must be that $b \geq \alpha(b)$, so $B \geq B'$.

It is sufficient to show that for any implementable mechanism (x', t') for the pair of value distributions (G_b, F_s), there exists an implementable mechanism (x, t) for the pair of value distributions (F_b, F_s) that generates at least the same amount of welfare as (x', t') under (G_b, F_s). Fix such a mechanism (x', t'). By Theorem 1 of Myerson and Satterthwaite (1983), $\bar{x}'_b$ and $\bar{x}'_s$ are increasing functions.

Define $x(b, s) = x'(\alpha(b), s)$, so that $x(B, S) = x'(B', s)$. We have that

$$\bar{x}_b(B) = \mathbb{E}\,[x(B, S)|B] = \mathbb{E}\left[x'(B', S)|B\right] = \bar{x}'_b(B').$$

Since α and $\bar{x}'_b$ are increasing functions, $\bar{x}_b$ must also be increasing. Similarly, $\bar{x}_s(S) = \bar{x}'_s(S)$ and $\bar{x}_s$ is decreasing (under the new notation, like in Section 5, $\bar{x}_s(s)$ denotes the probability that seller type s trades the good).

As (x', t') is implementable for the pair of distributions (G_b, F_s), Theorem 1 of Myerson and Satterthwaite (1983) implies that

$$\mathbb{E}\left[\bar{x}'_b(B')\psi_b^{G_b}(B')\right] \geq \mathbb{E}\left[\bar{x}_s(S)\psi_s^{F_s}(S)\right].$$

Then, the hypothesis that $\psi_b^{F_b}(B) \geq \psi_b^{G_b}(\alpha(B)) = \psi_b^{G_b}(B')$, along with $\bar{x}_b(B) = \bar{x}'_b(B')$, leads to

$$\mathbb{E}\left[\bar{x}_b(B)\psi_b^{F_b}(B)\right] \geq \mathbb{E}\left[\bar{x}_s(S)\psi_s^{F_s}(S)\right].$$

By Theorem 1 of Myerson and Satterthwaite, the inequality above along with the monotonicity of $\bar{x}_b$ and $\bar{x}_s$ implies the existence of a transfer function t such that (x, t) is an implementable mechanism for the pair of value distributions (F_b, F_s).

The difference in the welfare generated by (x, t) under (F_b, F_s) and by (x', t') under (G_b, F_s) is

$$\mathbb{E}\,[\bar{x}_b(B)B] - \mathbb{E}\left[\bar{x}'_b(B')B'\right] = \mathbb{E}\left[\bar{x}_b(B)(B - B')\right] \geq 0,$$

which completes the proof. □

Proof of Proposition 9. Let B and S be random variables with distributions F_b and F_s, respectively. Then, $S' := \alpha(S)$ is a random variable distributed according to G_s. It is sufficient to show that for any implementable mechanism (x', t') for the pair of value distributions (F_b, G_s) that obeys the seller's damage participation constraint for the value reduction from F_s to G_s, there exists an implementable mechanism (x, t) for the pair of value distributions (F_b, F_s) that generates at least the same amount of welfare as (x', t') under (F_b, G_s). Fix such a mechanism (x', t'), and define $x(b, s) = x'(b, \alpha(s))$, so that $x(b, S) = x'(b, S')$. Arguments analogous to those for Proposition 8 establish that $S \geq S'$, $\bar{x}_b(B) = \bar{x}'_b(B)$ and $\bar{x}_s(S) = \bar{x}'_s(S')$, and that $\bar{x}_b$ is increasing and $\bar{x}_s$ is decreasing.

The fact that (x', t') obeys the seller's damage participation constraint for the value reduction from F_s to G_s implies that the expected gains from trade of the highest value type $\alpha(\bar{s})$ under (x', t') are at least $\bar{s} - \alpha(\bar{s})$. As (x', t') is implementable for the pair of distributions (F_b, G_s), Theorem 1 of Myerson and Satterthwaite (1983) implies that

$$\mathbb{E}\left[\bar{x}'_b(B)\psi_b^{F_b}(B)\right] - \mathbb{E}\left[\bar{x}'_s(S')\psi_s^{G_s}(S')\right] \geq \bar{s} - \alpha(\bar{s}).$$

The hypothesis that $\psi_s^{F_s}(s) - \psi_s^{G_s}(\alpha(s)) \leq \bar{s} - \alpha(\bar{s})$ for all s in the support of F_s, along with $\bar{s} \geq \alpha(\bar{s})$ and $\bar{x}_s(S) = \bar{x}'_s(S') \in [0, 1]$, implies that $\bar{x}'_s(S')\psi_s^{G_s}(S') \geq \bar{x}_s(S)\psi_s^{F_s}(S) - (\bar{s} - \alpha(\bar{s}))$. Since $\bar{x}_b(B) = \bar{x}'_b(B)$, it follows that

$$\mathbb{E}\left[\bar{x}_b(B)\psi_b^{F_b}(B)\right] - \mathbb{E}\left[\bar{x}_s(S)\psi_s^{F_s}(S)\right] \geq 0.$$

As in the proof of Proposition 8, we conclude that there exists a transfer function t such that (x, t) is an implementable mechanism for the pair of value distributions (F_b, F_s). The difference in welfare generated by (x, t) under (F_b, F_s) and by (x', t') under (F_b, G_s) is $\mathbb{E}\left[(1 - \bar{x}_s(S))(S - S')\right] \geq 0$. □

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