

# WITHHOLDING AND DAMAGE IN BAYESIAN TRADE MECHANISMS

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**ABSTRACT.** We study the welfare consequences of allowing the mechanism designer to withhold or damage resources in the optimal mechanism for bilateral trade with independent private values. When the buyer and the seller have discrete value distributions, we establish the following results. Burning money or withholding the good from both traders never enhances welfare. Similarly, damaging the good for the buyer cannot improve welfare. By contrast, damaging the good for the seller may improve welfare in the optimal mechanism. However, such welfare improvements are possible only if the damage hurts some lower valuation type of seller more severely than the highest valuation type. Results extend to the case of continuous value distributions under additional hypotheses regarding virtual values. Methods also apply to optimal Bayesian implementation for allocation problems. In the absence of property rights, damaging goods for any agent has negative welfare consequences. The solution concepts of Bayesian and dominant strategy implementation have distinct implications for the optimality of burning money and withholding goods in these environments.

*Keywords:* mechanism design; Bayesian optimal mechanism; bilateral trade; allocation problems; money burning; withholding; damage.

## 1. INTRODUCTION

A major theme of the literature on mechanism design with asymmetric information is that to achieve an optimal allocation of resources, it may be necessary to *deliberately* introduce distortions that would be inefficient in a complete information setting. That is, it is optimal to violate *first-best* efficiency in order to relax incentive and individual rationality constraints.

For example, consider a seller who has an indivisible good to which she attaches no value facing a buyer whose valuation for the good is equally likely to be 3 or 7. In the mechanism that maximizes expected revenue, the seller sets a price of 7, and the buyer acquires the good when his valuation is 7, but not when it is 3. This outcome is inefficient because either buyer type derives greater value from the good than the seller. The inefficiency arises because by not trading with the low-value buyer, the seller is able to demand a high price from the high-value buyer. If the valuation of the high-type buyer were reduced from 7 to 5, then the revenue maximizing mechanism prescribes that the seller trade with both buyer types at price 3. The reduction in the value of the high-type buyer not only increases the probability of an efficient allocation under the revenue maximizing mechanism from  $1/2$  to 1, but also increases total expected welfare (the sum of expected buyer and seller payoffs) from  $7/2$  to 4.

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Going a step further, Green and Laffont (1979), Laffont and Maskin (1979) and Myerson and Satterthwaite (1983) exhibit models in which, because of asymmetric information, there is *no* mechanism that is first-best efficient and meets desirable individual rationality and budget balance criteria.<sup>1</sup> For an illustration, consider the following instance of the Myerson-Satterthwaite model of bilateral trade with incomplete information. A seller owns an indivisible good, which she values at either  $s_1 \in (0, 1)$  or 3 with equal probability. A buyer is interested in acquiring the good, and values it at either 1 or 4, also with equal probability. The valuations of the seller and the buyer are independently distributed. First-best efficiency dictates that the good be sold with probability 1 for the pairs of buyer and seller types  $(1, s_1)$ ,  $(4, s_1)$ ,  $(4, 3)$  and with probability 0 for the pair  $(1, 3)$ . However, there is no individually rational and Bayesian incentive compatible mechanism that delivers these outcomes. The *second-best* welfare optimal mechanism entails that for the pair of types  $(1, s_1)$ , the good is traded with probability  $2/(2 + s_1)$ . In this example, reducing the value  $s_1$  of the low-type seller increases the probability that the good is allocated efficiently in the optimal mechanism, and also enhances total welfare if  $s_1 \leq \sqrt{6} - 2$ .

The examples above suggest that reducing valuations and, more generally, creating inefficiencies at the ex ante stage of a mechanism may improve ex post allocative efficiency and social welfare. In this paper, we explore the possibility of such welfare improvements in the Myerson-Satterthwaite bilateral trade setting when the mechanism designer has access to two types of inefficient actions: withholding and damage.

Under *withholding*, we imagine that when the good is handed over by the seller, it does not necessarily reach the buyer: the mechanism designer can withhold or destroy the good with some probability. Similarly, the mechanism designer has the option to withhold part of the monetary transfer from the buyer to the seller (i.e., “burn money” or run an ex post budget surplus). We assume that the mechanism designer can condition both types of withholding on traders’ reports.

When the good is deliberately *damaged*, it is reduced in value for one or more buyer or seller types. This may entail physical damage that is intentional or exogenous. Regulations that restrict the use of the good or prohibit bundling may also reduce a trader’s valuation and have idiosyncratic effects on types (whereas property taxes reduce valuations uniformly across types). For a concrete example, suppose that the good being sold is a private airplane that has a navigational device attuned to the seller’s geographic region. The buyer lives in a different region, and removing the device will not affect him at all. However, removing it will impact a seller with low valuation more because higher-valuation seller types are intensely invested in flying and have other equipment that could substitute for the missing device. Similarly, a broker handling an estate or an art collection can decide which items should be included for sale as a package, and dictate what the seller should do with the excluded items prior to the sale.

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<sup>1</sup>As we discuss later, the Myerson-Satterthwaite impossibility theorem holds for continuous value distributions, but not for discrete ones.

We begin our analysis with the case in which the two traders have discrete value distributions. This setting allows us to develop sharp intuitions and to provide the strongest versions of the results. The general principle behind creating inefficiencies is that they may enhance a mechanism’s ability to discriminate between different agent types. Since our model (as much of the mechanism design literature) assumes quasilinear preferences, there is no difference between types in their marginal utility for money. Hence, there should be no welfare gain from withholding money, and this intuition is borne out by our Proposition 1.

More interestingly, Proposition 2 establishes that withholding the good never improves welfare either. To understand the issues involved in establishing this seemingly intuitive result, note that if a mechanism withholds the good for some pair of buyer and seller types, the natural path to welfare improvement requires increasing the probability that the good is allocated to either of these types. However, this perturbation may result in a violation of the monotonicity of traders’ probability of being allocated the good with respect to their types, which is a necessary condition for implementation. In this situation, we show how the allocation can be further perturbed via a sequence of changes in trading probabilities that has a neutral effect on the overall probability that each buyer and seller type affected by the changes is allocated the good until we eventually reach a type for which the monotonicity condition is not binding. With suitable adjustments in monetary transfers, the perturbed allocation is implementable and improves welfare over the original one.

Proposition 3 shows that damaging the good for the buyer is never socially optimal. The intuition is that reducing buyer valuations entails less scope for allocating the good to the buyer and thus decreases potential gains from trade.

By contrast, damaging the good for the seller may enhance social welfare, as our second opening example demonstrates. This is true even if the seller needs to consent to the damage, which imposes the constraint that the utility she obtains from participating in the mechanism should be greater than or equal to her value for the *undamaged* good. Proposition 4 synthesizes the potential welfare benefits of damage for the seller. The idea is that a seller type whose valuation has been reduced finds it less attractive to pretend to be a higher-type seller. The slack created in the underlying incentive constraint can be leveraged to increase the probability of trade for seller types with reduced valuations. Proposition 4 suggests that it may be valuable to employ a mechanism designer who is more “active” than usually considered.

However, Proposition 4 relies on the mechanism designer damaging the good to a greater extent for seller types that value it less. In some situations, the mechanism designer may be constrained to damage the good in a fashion that affects higher-valuation seller types equally or more. Then, Proposition 5 reverses the conclusion of Proposition 4 under the maintained assumption that the seller should weakly prefer trading the damaged good in the mechanism to walking away with the undamaged good: damage that affects the seller type with the highest valuation most severely cannot improve welfare. The proof of Proposition 5 shows that even though reduced valuations for lower seller types decrease the information rents commanded by the seller as highlighted in Proposition 4, the transfers the designer saves on lower types

by damaging the good are not sufficient to compensate the highest valuation seller type for participating in the mechanism and running the risk of being allocated the damaged good.

We develop parallel results for the original Myerson-Satterthwaite setting in which the buyer and the seller have continuous value distributions. Propositions 6, 7, 8 and 9 extend Propositions 1, 2, 3 and 5, respectively, under mild conditions regarding virtual valuations. We also briefly consider related questions for social choice rules and allocation problems without initial ownership as modeled by Wolitzky (2016).

There are several contributions related to our paper. Cramton et al. (1987) show that the tension between incomplete information and allocative efficiency highlighted by the impossibility theorem of Myerson and Satterthwaite is remediated by dispersing property rights among agents. Matsuo (1989) and Kos and Manea (2009) dispense with another key ingredient for the Myerson-Satterthwaite impossibility theorem—the assumption that agents have continuous value distributions—and characterize the discrete value distributions for which an ex post efficient allocation is implementable. Makowski and Mezzetti (1993) argue that the impossibility theorem is not robust to the introduction of a second buyer.

Guo and Conitzer (2008, 2009), Moulin (2009) and de Clippel et al. (2014) show that burning money and withholding goods enhance welfare in multi-unit allocation problems where agents do not have property rights.<sup>2</sup> Results in this literature are driven by the adoption of the stronger solution concept of dominant strategy incentive compatibility coupled with the requirement of no ex post budget deficit. By contrast, our results show that burning money and withholding goods cannot improve welfare in the Bayesian framework for bilateral trade. Remarkably, Drexel and Kleiner (2015) prove that in the absence of property rights, ex post efficient allocations are implementable in the Bayesian framework when agents have regular value distributions, closing the door on any potential benefits from withholding.<sup>3</sup> Our results regarding the negative welfare effects of reducing an agent’s valuation in allocation problems mirror the analysis of damage for the buyer in bilateral trade. However, the ownership structure in the bilateral trade setting generates fundamental asymmetries between the two traders that are not encountered in allocation problems, as demonstrated by the distinct welfare effects of damaging the good for the buyer and for the seller.

The effects of withholding and damaging goods have been considered in other strategic environments by Postlewaite (1979), Deneckere and McAfee (1996), Hart and Reny (2015) and Condorelli and Szentes (2019).

The rest of the paper is organized as follows. In the next section, we set up the model with discrete types. Sections 3 and 4 present the results for money burning and withholding the good, respectively. In Sections 5 and 6, we analyze the effects of damaging the good for the

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<sup>2</sup>Other related contributions include Hartline and Roughgarden (2008) and Long et al. (2017).

<sup>3</sup>Green and Laffont (1979) showed that ex post efficiency cannot be achieved under dominant strategy implementation subject to ex post budget balance. Drexel and Kleiner note that Green and Laffont’s conclusion is reversed if the budget balance constraint is relaxed from the ex post to the ex ante stage. Drexel and Kleiner also cogently argue that the results of Manelli and Vincent (2010) and Gershkov et al. (2013) regarding the equivalence of Bayesian and dominant strategy implementation in general classes of mechanism design problems do not apply under the additional desideratum of no ex post budget deficit.

buyer and for the seller, respectively. Section 7 extends our results to the continuous-type setting. In Section 8, we discuss mechanisms for allocation problems without initial ownership. Section 9 provides concluding remarks. Proofs are relegated to the Appendix.

## 2. FRAMEWORK

We first consider a discrete-value version of the bilateral trade problem with incomplete information modeled by Myerson and Satterthwaite (1983), in which a *buyer* is interested in acquiring an indivisible *good* that a *seller* owns. The *valuations* of the buyer and of the seller for the good are independently distributed random variables with *probability mass functions*  $p_b$  and  $p_s$ , respectively, which have corresponding *supports*  $V_b = \{b_1, b_2, \dots, b_m\}$  and  $V_s = \{s_1, s_2, \dots, s_n\}$ , where  $0 \leq b_1 < b_2 < \dots < b_m$  and  $0 \leq s_1 < s_2 < \dots < s_n$ . Each trader is privately informed about his or her own value for the good (the trader's *type*) and believes that the other trader's value is a random variable drawn from the specified distribution. The two traders are risk neutral and have additively separable utility functions for money and the good.

The mechanism designer needs to specify a game that determines the probability of allocating the good to each of the two traders and the monetary transfers sent or received by the traders. The objective of the mechanism designer is to maximize expected total welfare in some Bayesian Nash equilibrium of the game. By the revelation principle, we can focus the analysis on direct mechanisms without loss of generality. In a *direct mechanism*, the traders simultaneously report their values, and the outcome is determined by four functions  $(x_b, x_s, t_b, t_s)$  with  $x_b, x_s : V_b \times V_s \rightarrow [0, 1]$  and  $t_b, t_s : V_b \times V_s \rightarrow \mathbb{R}$ . If the buyer reports value  $b_i$  and the seller reports value  $s_j$ , then  $x_b(b_i, s_j)$  and  $x_s(b_i, s_j)$  are the probabilities with which the good is *allocated* to the buyer and to the seller, respectively, and  $t_b(b_i, s_j)$  is the *monetary transfer sent* by the buyer, whereas  $t_s(b_i, s_j)$  is the *monetary transfer received* by the seller.

This formulation departs from the Myerson-Satterthwaite model by allowing the mechanism designer to withhold both money and the good: for a profile of reports  $(b_i, s_j)$ ,  $t_b(b_i, s_j) - t_s(b_i, s_j)$  is the amount of money withheld (or “burned”), and  $1 - x_b(b_i, s_j) - x_s(b_i, s_j)$  is the probability that the good is withheld (or destroyed) by the mechanism designer. The mechanism designer does not bring any resources to the market, and *feasibility* requires that  $t_b(b_i, s_j) - t_s(b_i, s_j) \geq 0$  and  $x_b(b_i, s_j) + x_s(b_i, s_j) \leq 1$ .

A direct mechanism is (Bayesian) *incentive compatible* if honest reporting of values forms a Bayesian Nash equilibrium of the game between the buyer and the seller induced by the mechanism. A mechanism is (interim) *individually rational* if each trader type obtains non-negative expected utility gains from participating in the mechanism. We say that a mechanism  $(x_b, x_s, t_b, t_s)$  is *implementable* if it is feasible, individually rational, and incentive compatible. An allocation  $(x_b, x_s)$  is *implementable* if there exist transfer functions  $(t_b, t_s)$  such that  $(x_b, x_s, t_b, t_s)$  is an implementable mechanism. The designer's objective is to specify an implementable mechanism that maximizes *total welfare*—the sum of expected utility for the buyer and the seller.

For a mechanism  $(x_b, x_s, t_b, t_s)$ , let

$$\bar{x}_b(b_i) = \sum_{j=1}^n p_s(s_j) x_b(b_i, s_j) \ \& \ \bar{t}_b(b_i) = \sum_{j=1}^n p_s(s_j) t_b(b_i, s_j)$$

denote the probability that the buyer of type  $b_i$  receives the good and the expected payment type  $b_i$  makes respectively, and let

$$\bar{x}_s(s_j) = \sum_{i=1}^m p_b(b_i) x_s(b_i, s_j) \ \& \ \bar{t}_s(s_j) = \sum_{i=1}^m p_b(b_i) t_s(b_i, s_j)$$

denote the probability that the seller of type  $s_j$  keeps the good and the expected payment type  $s_j$  receives, respectively. An *optimal implementable mechanism*  $(x_b, x_s, t_b, t_s)$  solves the following linear program:<sup>4</sup>

$$\begin{aligned} \max \quad & \sum_{i=1}^m p_b(b_i) (\bar{x}_b(b_i) b_i - \bar{t}_b(b_i)) + \sum_{j=1}^n p_s(s_j) (\bar{x}_s(s_j) s_j + \bar{t}_s(s_j)) \\ \text{s.t.} \quad & IR_{b_i} : \bar{x}_b(b_i) b_i - \bar{t}_b(b_i) \geq 0, \forall i = \overline{1, m} \\ & IR_{s_j} : \bar{x}_s(s_j) s_j + \bar{t}_s(s_j) \geq s_j, \forall j = \overline{1, n} \\ & IC_{b_i \rightarrow b_k} : \bar{x}_b(b_i) b_i - \bar{t}_b(b_i) \geq \bar{x}_b(b_k) b_k - \bar{t}_b(b_k), \forall i, k = \overline{1, m} \\ & IC_{s_j \rightarrow s_k} : \bar{t}_s(s_j) - (1 - \bar{x}_s(s_j)) s_j \geq \bar{t}_s(s_k) - (1 - \bar{x}_s(s_k)) s_k, \forall j, k = \overline{1, n} \\ & FT_{b_i, s_j} : t_b(b_i, s_j) - t_s(b_i, s_j) \geq 0, \forall i = \overline{1, m}, j = \overline{1, n} \\ & FX_{b_i, s_j} : x_b(b_i, s_j) + x_s(b_i, s_j) \leq 1, \forall i = \overline{1, m}, j = \overline{1, n}. \end{aligned}$$

The seller's individual rationality constraint  $IR_{s_j}$  guarantees that by participating in the mechanism, a seller of type  $s_j$  obtains utility at least  $s_j$ , reflecting the seller's initial ownership of the good (and option to walk away from the mechanism and hold on to the good).

Myerson and Satterthwaite (1983) assume that the mechanism designer cannot withhold either money or the good, which corresponds to imposing the feasibility constraints  $FT_{b_i, s_j}$  and  $FX_{b_i, s_j}$  with equality for all  $i$  and  $j$ . This assumption restricts the set of mechanisms the designer can implement. The next two sections investigate whether giving the mechanism designer the freedom to withhold money or the good can enhance welfare.

### 3. MONEY BURNING

We first show that withholding monetary transfers (“burning money”) is never optimal.

**Proposition 1.** *Optimal implementable mechanisms never withhold monetary transfers.*

The proof of this and subsequent results are presented in the Appendix. The argument is straightforward. If an implementable mechanism involves money withholding for a pair of reports  $(b_i, s_j)$ , we can increase the transfer to seller type  $s_j$  to match the transfer from  $b_i$  when the buyer reports type  $b_i$ , and reduce the transfers  $s_j$  receives from all buyer types uniformly and then credit them to the buyer, so that  $s_j$  enjoys the same expected outcomes

<sup>4</sup>The notation  $\overline{a, b}$  stands for the sequence of integers in the interval  $[a, b]$ .

following the perturbation. The perturbed mechanism preserves individual rationality and incentive compatibility, and increases the expected utility of the buyer without affecting the expected utility of the seller.

#### 4. WITHHOLDING THE GOOD

The previous section shows that in solving the linear program for the optimal mechanism, we can assume that the  $FT_{b_i, s_j}$  constraints hold with equality, i.e.,  $t_b(b_i, s_j) = t_s(b_i, s_j)$  for all  $(b_i, s_j)$ . In what follows, we restrict attention to mechanisms with this property, and use a single function  $t$  to describe monetary transfers; notation for a mechanism simplifies to  $(x_b, x_s, t)$  with the understanding that  $t_b(b_i, s_j) = t_s(b_i, s_j) = t(b_i, s_j)$ .

The next result shows that withholding the good cannot enhance welfare under the optimal mechanism.

**Proposition 2.** *There exists an optimal implementable mechanism that withholds the good with probability zero. If  $s_1 > 0$ , then every optimal implementable mechanism withholds the good with probability zero.*

The following lemmata collect standard insights from mechanism design about binding “local” incentive constraints and monotonicity properties of the allocation needed for the proof (some appear in Lovejoy 2006).

**Lemma 1.** *A mechanism  $(x_b, x_s, t)$  is implementable if and only if it satisfies  $IR_{b_1}$ ,  $IR_{s_n}$ ,  $IC_{b_i \rightarrow b_{i+1}}$ ,  $IC_{b_{i+1} \rightarrow b_i}$ ,  $IC_{s_j \rightarrow s_{j+1}}$  and  $IC_{s_{j+1} \rightarrow s_j}$  for  $i = \overline{1, m-1}$  and  $j = \overline{1, n-1}$ . If  $(x_b, x_s, t)$  is implementable, then  $\bar{x}_b(b_{i+1}) \geq \bar{x}_b(b_i)$  and  $\bar{x}_s(s_{j+1}) \geq \bar{x}_s(s_j)$  for  $i = \overline{1, m-1}$  and  $j = \overline{1, n-1}$ . For any mechanism  $(x_b, x_s, t)$ , if  $\bar{x}_b(b_{i+1}) > \bar{x}_b(b_i)$  and  $IC_{b_{i+1} \rightarrow b_i}$  holds with equality, then  $IC_{b_i \rightarrow b_{i+1}}$  is satisfied with strict inequality. Similarly, if  $\bar{x}_s(s_{j+1}) > \bar{x}_s(s_j)$  and  $IC_{s_j \rightarrow s_{j+1}}$  holds with equality, then  $IC_{s_{j+1} \rightarrow s_j}$  is satisfied with strict inequality.*

**Lemma 2.** *For every implementable mechanism  $(x_b, x_s, t)$ , there exists a transfer specification  $t'$  such that  $(x_b, x_s, t')$  is implementable, and  $IR_{b_1}$ ,  $IC_{b_i \rightarrow b_{i-1}}$  and  $IC_{s_j \rightarrow s_{j+1}}$  hold with equality under  $(x_b, x_s, t')$  for  $i = \overline{2, m}$  and  $j = \overline{1, n-1}$ .*

We sketch the proof for the second part of Proposition 2 here (the first part follows from similar ideas and a compactness argument). Suppose that  $s_1 > 0$ , and consider an implementable mechanism  $(x_b, x_s, t'')$  that withholds the good with positive probability for a profile of reports  $(b_i, s_j)$ , i.e.,  $x_b(b_i, s_j) + x_s(b_i, s_j) < 1$ . Lemma 2 implies the existence of a transfer function  $t$  that implements the same allocation  $(x_b, x_s)$  such that  $(x_b, x_s, t)$  satisfies the constraints listed in the lemma with equality. Since neither mechanism withholds money,  $(x_b, x_s, t)$  generates the same total welfare as  $(x_b, x_s, t'')$ , so  $(x_b, x_s, t'')$  is optimal if and only if  $(x_b, x_s, t)$  is optimal. To increase the welfare generated by  $(x_b, x_s, t)$ , we contemplate a perturbation of the allocation whereby the seller of type  $s_j$  keeps the good with higher probability in the event agents report types  $(b_i, s_j)$ :  $x'_s(b_i, s_j) = x_s(b_i, s_j) + \varepsilon$ .

If  $\bar{x}_s(s_j) < \bar{x}_s(s_{j+1})$  and  $\varepsilon$  is sufficiently small, the monotonicity of  $\bar{x}_s$  implies that  $\bar{x}'_s$  is monotonic, which per Lemma 1 is a necessary condition for the implementation of the

allocation  $(x_b, x'_s)$ . To provide incentives supporting the new allocation  $(x_b, x'_s)$ , we modify transfers to type  $s_j$  uniformly so that  $s_j$  receives the same utility in the perturbed mechanism:  $t'(b_k, s_j) = t(b_k, s_j) - p_b(b_i)\varepsilon s_j$  for all  $k$ .  $IC_{s_{j+1} \rightarrow s_j}$  is the main constraint from Lemma 1 we need to check in order to establish that  $(x_b, x'_s, t')$  is an implementable mechanism. Indeed, the perturbation reduces payments to a type  $s_j$  by  $p_b(b_i)\varepsilon s_j$  to account for the extra probability  $p_b(b_i)\varepsilon$  with which this type is allocated the good. This marginal change is also attractive to type  $s_{j+1}$  for whom the increase in the allocation probability is worth  $p_b(b_i)\varepsilon s_{j+1}$ , which is greater than the implicit cost of  $p_b(b_i)\varepsilon s_j$ . However, the condition  $\bar{x}_s(s_j) < \bar{x}_s(s_{j+1})$  and the assumption that  $(x_b, x_s, t)$  satisfies  $IC_{s_j \rightarrow s_{j+1}}$  with equality, along with Lemma 1, imply that  $(x_b, x_s, t)$  satisfies  $IC_{s_{j+1} \rightarrow s_j}$  with strict inequality. By continuity,  $(x_b, x'_s, t')$  must also satisfy  $IC_{s_{j+1} \rightarrow s_j}$  for small  $\varepsilon$ . The perturbation increases welfare by  $p_b(b_i)p_s(s_j)\varepsilon s_j > 0$ , which means that  $(x_b, x_s, t)$  is suboptimal.

The case  $\bar{x}_s(s_j) = \bar{x}_s(s_{j+1})$  requires a more extensive sequence of perturbations since the monotonicity condition necessary for implementability from Lemma 1 cannot be maintained if we increase  $\bar{x}_s(s_j)$  without altering  $\bar{x}_s(s_{j+1})$ . However, in seeking a welfare improvement over the mechanism  $(x_b, x_s, t)$ , we still take advantage of the slack in the constraint  $FX_{b_i, s_j}$  by increasing the probability that type  $s_j$  keeps the good to  $x'_s(b_i, s_j) = x_s(b_i, s_j) + \varepsilon$  for some small  $\varepsilon$ . For this change to have a neutral effect on  $\bar{x}_s(s_j)$ , we decrease  $x_s(b_{i'}, s_j)$  and increase  $x_b(b_{i'}, s_j)$  by  $\varepsilon p_b(b_i)/p_b(b_{i'})$  for some other buyer type  $b_{i'}$ . We can then neutralize the effect of this perturbation on type  $b_{i'}$  by decreasing the probability that  $b_{i'}$  receives the good from some other seller type  $s_{j'}$ . Finally, the increase in the probability that  $s_{j'}$  keeps the good can be implemented without a further change in the allocation of  $s_{j'}$  using the idea from the first step of the proof if  $\bar{x}_s(s_{j'}) < \bar{x}_s(s_{j'+1})$ . In the proof, we choose  $s_{j'}$  to be the highest seller type such that  $\bar{x}_s(s_{j'}) = \bar{x}_s(s_j)$ , and demonstrate the existence of a buyer type  $b_{i'}$  such that all perturbations generate allocation probabilities in  $[0, 1]$ . We then adjust transfers for seller type  $s_{j'}$  as in the first step of the proof to preserve the structure of incentives for the seller. We show that the perturbed mechanism boosts the welfare of  $(x_b, x_s, t)$  by  $p_b(b_i)p_s(s_j)\varepsilon s_{j'} > 0$ , proving that  $(x_b, x_s, t)$  is not optimal.

Interestingly, an analogous argument does not go through if  $\bar{x}_b(b_i) < \bar{x}_b(b_{i+1})$ , and we perturb the mechanism  $(x_b, x_s, t)$  to allocate the good with  $\varepsilon$  additional probability to the buyer—rather than to the seller—in the event agents report types  $(b_i, s_j)$ , i.e.,  $x'_b(b_i, s_j) = x_b(b_i, s_j) + \varepsilon$ . Indeed, if  $IC_{b_{i+1} \rightarrow b_i}$  and  $IC_{b_i \rightarrow b_{i-1}}$  are binding under  $(x_b, x_s, t)$  (as would happen under any optimal mechanism that does not implement an ex post efficient allocation),<sup>5</sup> then maintaining these constraints under the perturbed mechanism  $(x'_b, x'_s, t')$  requires that  $p_s(s_j)\varepsilon b_{i+1} \leq \bar{t}'_b(b_i) - \bar{t}_b(b_i) \leq p_s(s_j)\varepsilon b_i$ , which is impossible because  $p_s(s_j) > 0$  and  $b_{i+1} > b_i$ . This explains the buyer-seller asymmetry in the statement of Proposition 2. The forthcoming analysis of damaging the good for the buyer and for the seller reveals other asymmetries in the interaction between the allocation and incentives for the two agents.

<sup>5</sup>This follows from the same ideas as Lemma 3 in Section 6, which provides necessary conditions for optimal mechanisms under no withholding.

## 5. DAMAGING THE GOOD FOR THE BUYER

In this section and the next, we examine whether damaging the good for the buyer or for the seller can enhance welfare. Specifically, we assume that the mechanism designer can reduce the value of either the buyer or the seller for the good by shifting the corresponding value distribution down in the sense of first-order stochastic dominance. The shift does not have to be uniform across agent types. If the good has multiple components (or is offered as a bundle) and there is heterogeneity in how agent types value each component, then the mechanism designer may “damage” the good by prohibiting the sale or regulating the use of certain components with varying effects on different types. Uniform value reduction can also be accomplished by imposing a property tax.

Since Propositions 1 and 2 show that withholding money and the good cannot improve welfare, we revert to the Myerson-Satterthwaite formulation of the optimal mechanism in which the constraints  $FT_{b_i, s_j}$  and  $FX_{b_i, s_j}$  are imposed with equality. Following the notation from the previous section, the *transfer* function  $t : V_b \times V_s \rightarrow \mathbb{R}$  specifies a payment from the buyer to the seller. We *update notation* for allocations to reflect the fact that withholding the good is ruled out: the *allocation* is now determined by a single function  $x : V_b \times V_s \rightarrow [0, 1]$  that describes the *probability of trade* between the buyer and the seller. In the *mechanism*  $(x, t)$ , when the buyer reports type  $b_i$  and the seller reports type  $s_j$ , the two agents trade the good with probability  $x(b_i, s_j)$  in exchange for a payment  $t(b_i, s_j)$ .

Let

$$\bar{x}_b(b_i) = \sum_{j=1}^n p_s(s_j)x(b_i, s_j) \ \& \ \bar{t}_b(b_i) = \sum_{j=1}^n p_s(s_j)t(b_i, s_j)$$

denote the probability that the buyer of type  $b_i$  *buys* the good and the expected payment type  $b_i$  *sends* to the seller, respectively, and let

$$\bar{x}_s(s_j) = \sum_{i=1}^m p_b(b_i)x(b_i, s_j) \ \& \ \bar{t}_s(s_j) = \sum_{i=1}^m p_b(b_i)t(b_i, s_j)$$

denote the probability that the seller of type  $s_j$  *sells* the good and the expected payment type  $s_j$  *receives* from the buyer, respectively (note the *change in notation*: unlike in the previous sections,  $\bar{x}_s$  denotes trading probabilities for the seller here). The optimal implementable mechanism  $(x, t)$  then solves the following linear program:

$$\begin{aligned} \max \quad & \sum_{i=1}^m p_b(b_i)\bar{x}_b(b_i)b_i + \sum_{j=1}^n p_s(s_j)(1 - \bar{x}_s(s_j))s_j \\ \text{s.t.} \quad & IR_{b_i} : \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) \geq 0, \forall i = \overline{1, m} \\ & IR_{s_j} : \bar{t}_s(s_j) - \bar{x}_s(s_j)s_j \geq 0, \forall j = \overline{1, n} \\ & IC_{b_i \rightarrow b_k} : \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) \geq \bar{x}_b(b_k)b_i - \bar{t}_b(b_k), \forall i, k = \overline{1, m} \\ & IC_{s_j \rightarrow s_k} : \bar{t}_s(s_j) - \bar{x}_s(s_j)s_j \geq \bar{t}_s(s_k) - \bar{x}_s(s_k)s_j, \forall j, k = \overline{1, n}. \end{aligned}$$

We establish that damaging the good for the buyer can never improve welfare.

**Proposition 3.** *If  $p_b$  and  $p'_b$  are two value distributions for the buyer such that  $p_b$  first-order stochastically dominates  $p'_b$ , then for any seller value distribution  $p_s$ , no implementable mechanism for the pair of value distributions  $(p'_b, p_s)$  achieves greater total welfare than the optimal implementable mechanism for  $(p_b, p_s)$ .*

The intuition for this result is that reducing buyer value entails less scope for allocating the good to the buyer and thus decreases potential gains from trade. For the proof, it is helpful to approach this intuition from the opposite angle: if the buyer value distribution improves from  $p'_b$  to  $p_b$ , it becomes easier to incentivize trade with buyer types whose values increase.

We can assume that the optimal mechanisms for  $(p_b, p_s)$  and  $(p'_b, p_s)$  are defined for buyer valuations in the union of the supports of  $p_b$  and  $p'_b$  since any implementable mechanism can be extended to a set of probability-zero types by specifying allocations and transfers for each such type as an optimal selection for that type from the menu of allocations and transfers prescribed by the mechanism for positive-probability types, along with the option of no trade and zero transfers. For any pair of distributions with finite supports that is ranked with respect to first-order stochastic dominance, we can reach the dominating distribution from the dominated one via a finite sequence of changes that move probability mass up between single pairs of adjacent points in the union of the two supports. Therefore, it is sufficient to establish the result for the case in which  $p_b(b_i) = p'_b(b_i) - \delta$  and  $p_b(b_{i+1}) = p'_b(b_{i+1}) + \delta$  for some  $\delta > 0$ , and  $p'_b(b_k) = p_b(b_k)$  for  $k \neq i, i + 1$ .

We show that any mechanism  $(x', t')$  that is implementable when traders' values are distributed according to  $(p'_b, p_s)$  and satisfies the constraint  $IC_{b_{i+1} \rightarrow b_i}$  from Lemma 2 with equality can be perturbed to obtain an implementable mechanism  $(x, t)$  for the value distributions  $(p_b, p_s)$  that differs from  $(x', t')$  only for the following profiles of types:

$$\begin{aligned} x(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})} x'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})} x'(b_{i+1}, s_j) \\ t(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})} t'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})} t'(b_{i+1}, s_j), \forall j = \overline{1, n}. \end{aligned}$$

The perturbed mechanism  $(x, t)$  channels  $\delta \bar{x}'_b(b_i)$  volume of trade from the measure  $\delta$  of buyer types whose valuations improve from  $b_i$  to  $b_{i+1}$  in the shift from  $p'_b$  to  $p_b$  without affecting the expected pattern of trade for other buyer and seller types, and consequently generates  $\delta \bar{x}'_b(b_i)(b_{i+1} - b_i) \geq 0$  more surplus than  $(x', t')$ .

## 6. DAMAGING THE GOOD FOR THE SELLER

We now consider the situation in which the mechanism designer can damage the good for the seller. It is reasonable to assume that the seller has the option to keep the undamaged good, so the mechanism designer is constrained to provide each seller type at least the utility the seller would derive from consuming the good prior to the damage. To express this participation constraint for the seller, we focus on *type-by-type* damage whereby the value of each seller type  $j$  is reduced from  $s_j$  to  $s'_j$ . Thus, the mechanism designer can shift down the initial distribution of seller values  $p_s$  with support  $s_1 < s_2 < \dots < s_n$  to a distribution  $p'_s$  with

support  $s'_1 < s'_2 < \dots < s'_n$  such that  $s'_j \leq s_j$  and  $p_s(s_j) = p'_s(s'_j)$  for  $j = \overline{1, n}$ . Then, in the notation of the previous section, the *damage participation constraint* for seller type  $j$  in an implementable mechanism  $(x, t)$  following the value reduction from  $p_s$  to  $p'_s$  requires that

$$\bar{t}_s(s'_j) + (1 - \bar{x}_s(s'_j))s'_j \geq s_j.$$

Note that the standard  $IR_{s'_j}$  constraint for the pair of distributions  $(p_b, p'_s)$  is weaker, with  $s_j$  being replaced by  $s'_j$  on the right-hand side of the inequality above.

In contrast to Proposition 3, the next result shows that damaging the good for the seller may enhance welfare.

**Proposition 4.** *Let  $x$  be an implementable allocation for a pair of value distributions  $(p_b, p_s)$ . Suppose that there exist  $i \leq m - 1$  and  $j \leq n - 1$  such that  $\bar{x}_b(b_1) = \bar{x}_b(b_i) < \bar{x}_b(b_{i+1})$  and  $\bar{x}_s(s_1) = \bar{x}_s(s_j) > \bar{x}_s(s_{j+1}) > 0$ . If  $b_i > s_1$  and  $x(b_i, s_1) < 1$ , then for any sufficiently small  $\varepsilon > 0$ , damaging the good to reduce the valuations of seller types  $1, \dots, j$  uniformly by  $\varepsilon$  (without affecting other types) enables the mechanism designer to implement an allocation  $x'$  (while obeying the seller's damage participation constraint) that coincides with  $x$  for every pair of corresponding buyer and seller types except that*

$$x'(b_i, s_1 - \varepsilon) = x(b_i, s_1) + \varepsilon \frac{(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) \sum_{l=1}^j p_s(s_l)}{p_s(s_1)(b_{i+1} - b_i) \sum_{k=i+1}^m p_b(b_k)}.$$

*It is possible that the allocation  $x'$  generates greater total welfare given the damaged seller values than  $x$  given the original seller values even in situations where  $x$  is an optimal implementable allocation for  $(p_b, p_s)$ .<sup>6</sup>*

To understand this result, note that  $\bar{x}_s(s_1) = \bar{x}_s(s_j)$  implies that any mechanism implementing allocation  $x$  generates the same expected outcomes  $(\bar{x}_s$  and  $\bar{t}_s)$  for seller types 1 through  $j$ , and damaging the good for these types decreases their incentives to mimic higher types. Indeed, higher seller types keep the good with higher probability, but keeping the good becomes less attractive for a low-type seller whose value has been reduced. The slack created in the constraints  $IC_{s_{j'} \rightarrow s_{j+1}}$  by the  $\varepsilon$  reduction in valuations for seller types  $j' \leq j$  can be used to decrease transfers from the buyer of type  $i + 1$  and higher to the seller of type  $j$  and lower. This perturbation introduces slack in the constraint  $IC_{b_{i+1} \rightarrow b_i}$ , which can be leveraged to increase the probability of an ex post efficient trade between the seller with reduced valuation  $s_1 - \varepsilon$  and the buyer with valuation  $b_i$  at marginal terms that make type  $b_i$  indifferent and would be attractive to type  $b_{i+1}$  if not for the decrease in transfers granted to this type under the perturbation.

However, the improvement in allocative efficiency generated by increased trading between types  $b_i$  and  $s_1 - \varepsilon$  is achieved at the cost of damaging the good and thus reducing the utility of affected seller types in the event of no trade. Hence, there is a trade-off between the additional

<sup>6</sup>We can replace the hypothesis  $x(b_i, s_1) < 1$  with the weaker condition  $\bar{x}_s(s_1) < 1$ , and adjust the conclusion to state that  $\bar{x}'$  coincides with  $\bar{x}$  for all corresponding buyer and seller types except that  $\bar{x}'_b(b_i) = \bar{x}_b(b_i) + \varepsilon' p_s(s_1)$  and  $\bar{x}'_s(s_1 - \varepsilon) = \bar{x}_s(s_1) + \varepsilon' p_b(b_i)$  for some  $\varepsilon' > 0$ . See the Appendix for a proof.

gains from trade for the pair  $(b_i, s_1 - \varepsilon)$ ,

$$(1) \quad p_b(b_i)p_s(s_1)(b_i - s_1 + \varepsilon)(x'(b_i, s_1 - \varepsilon) - x(b_i, s_1)) \\ = \varepsilon \frac{(b_i - s_1 + \varepsilon)(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))p_b(b_i) \sum_{l=1}^j p_s(s_l)}{(b_{i+1} - b_i) \sum_{k=i+1}^m p_b(b_k)},$$

and the expected utility loss suffered by seller types 1 through  $j$  in the event that they do not trade the damaged good,

$$(2) \quad \varepsilon(1 - \bar{x}_s(s_1)) \sum_{l=1}^j p_s(s_l).$$

Both the marginal efficiency gain deriving from the perturbed allocation and the marginal loss resulting from the damage are of order  $\varepsilon$ , and the net effect of the perturbation on total welfare depends on the original allocation  $x$  and the value distributions  $(p_b, p_s)$ .

We illustrate this trade-off in an example showing that either effect (1) or (2) can dominate when  $x$  is the optimal implementable allocation for the original value distributions. Suppose that there are two buyer and two seller types ( $m = n = 2$ ). The buyer's valuations are  $b_1 = 1$  and  $b_2 = 4$  with equal probability, while the seller's valuations are (a parameter)  $s_1 \in [0, 1)$  and  $s_2 = 3$  with equal probability. We assume that the designer can damage the good for the low-type seller alone, and consider the welfare consequences of reducing  $s_1$ . The following lemma provides a method for computing the optimal implementable mechanism in this family of examples.

**Lemma 3.** *If  $(x, t)$  is an optimal implementable mechanism and  $x$  is not an ex post efficient allocation, then  $(x, t)$  satisfies all the constraints  $IR_{b_1}$ ,  $IR_{s_n}$ ,  $IC_{b_i \rightarrow b_{i-1}}$ , and  $IC_{s_j \rightarrow s_{j+1}}$  with equality for  $i = \overline{2, m}$  and  $j = \overline{1, n-1}$ .*

In the Appendix, we solve the relaxed linear program that determines the optimal implementable mechanism for the assumed value distributions in which constraints are replaced by equalities derived from the binding constraints  $IR_{b_1}$ ,  $IR_{s_2}$ ,  $IC_{b_2 \rightarrow b_1}$ , and  $IC_{s_1 \rightarrow s_2}$  identified in Lemma 3, and then argue that the solution indeed represents the optimal mechanism. The optimal mechanism implements the following allocation:

$$(3) \quad x(b_1, s_1) = 2/(2 + s_1), x(b_1, s_2) = 0, x(b_2, s_1) = 1, x(b_2, s_2) = 1.$$

The optimal mechanism allocates the good efficiently with higher probability as  $s_1$  decreases to 0, reaching ex post efficiency for  $s_1 = 0$ .<sup>7</sup> As implied by Proposition 4, making the good less valuable for the low-type seller  $s_1$  reduces her incentive to mimic the high type  $s_2$ , who trades less in the optimal mechanism. The resulting slack in the constraint  $IC_{s_1 \rightarrow s_2}$  is leveraged to increase the probability of trade between types  $b_1$  and  $s_1$ .

<sup>7</sup>Myerson and Satterthwaite (1983) note that their result about the impossibility of implementing the ex post efficient allocation for continuous value distributions does not extend to the case of discrete distributions. Our example with  $s_1 = 0$  is an instance where their impossibility result does not apply.

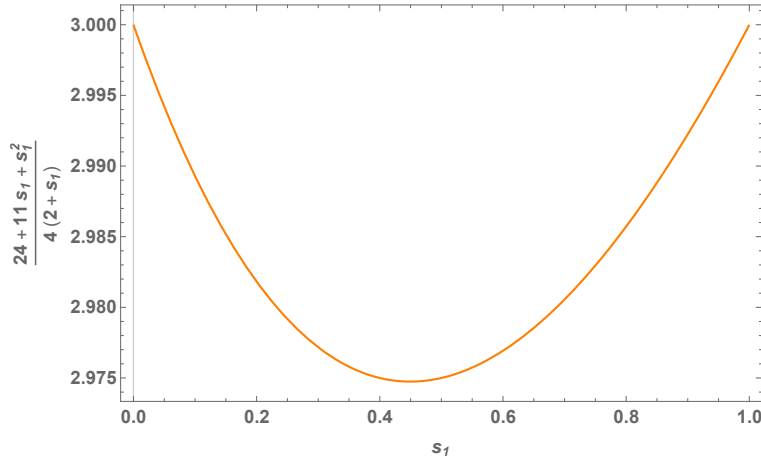


FIGURE 1. Total welfare in the optimal mechanism as a function of  $s_1$

The optimal implementable allocation  $x$  identified in (3) generates a total welfare of

$$\frac{24 + 11s_1 + s_1^2}{4(2 + s_1)},$$

which is decreasing for  $s_1 \in [0, \sqrt{6} - 2]$  and increasing for  $s_1 \in [\sqrt{6} - 2, 1)$  as seen in Figure 1. At  $s_1 = \sqrt{6} - 2 \approx 0.449$ , the marginal improvement in gains from trade quantified by formula (1) resulting from a small decrease in  $s_1$  exactly offsets the marginal utility loss described by expression (2) that seller type  $s_1$  suffers in the event she retains the damaged good. For  $s_1 < \sqrt{6} - 2$ , the gain is greater than the loss, while for  $s_1 > \sqrt{6} - 2$  the opposite is true. For a concrete computation, when  $s_1$  takes values 0 and 0.4, the corresponding optimal implementable mechanisms achieve total welfare 3 and 2.975, respectively. Therefore, reducing the value of the low-type seller from 0.4 to 0 enhances welfare under the optimal mechanism.

The seller's damage participation constraint is satisfied in this example when the designer damages the good for the low-type seller and implements the optimal mechanism given the seller's reduced valuation. Indeed, a seller with any initial valuation  $s_1 \in (0, 1)$  would consent to the damage and then participate in the optimal mechanism with a reduced valuation  $s'_1 \in [0, s_1)$  because trading in the optimal mechanism with value  $s'_1$  yields an expected payoff of at least  $3/2$ , which exceeds the seller's value  $s_1$  for the undamaged good. To see this, note that the individual rationality constraint  $IR_{s_2}$  and the incentive compatibility constraint  $IC_{s'_1 \rightarrow s_2}$  hold with equality under the optimal mechanism for the reduced value  $s'_1$ . Since seller type  $s_2$  trades the good with probability  $1/2$  under this mechanism, she receives an expected payment of  $1/2 \times s_2 = 3/2$ . Then, seller type  $s'_1$  expects a utility of  $1/2 \times s'_1 + 3/2 \geq 3/2$  under the mechanism.

Proposition 4 presumes that the mechanism designer is able to target the damage of the good at seller types with low valuations. Suppose instead that damage must hurt the seller type with the highest valuation most severely. Specifically, damage is restricted to shifting down the seller value distribution  $p_s$  with support  $s_1 < s_2 < \dots < s_n$  to distributions  $p'_s$  with support  $s'_1 < s'_2 < \dots < s'_n$  such that  $0 \leq s_j - s'_j \leq s_n - s'_n$  and  $p'_s(s'_j) = p_s(s_j)$  for  $j = \overline{1, n}$ .

In this case, we say that  $p'_s$  reflects damage at the top relative to  $p_s$ . We show that damage at the top for the seller cannot raise expected welfare.

**Proposition 5.** *If  $p'_s$  is a seller value distribution that reflects damage at the top relative to  $p_s$ , then for any buyer value distribution  $p_b$ , there is no implementable mechanism for the pair of value distributions  $(p_b, p'_s)$  that obeys the seller's damage participation constraint for the value reduction from  $p_s$  to  $p'_s$  and achieves greater total welfare than the optimal implementable mechanism for  $(p_b, p_s)$ .<sup>8</sup>*

To develop intuition for this result, it is helpful to revisit Proposition 4 and the binding constraints for the seller under optimal mechanisms from Lemma 3. As discussed in the context of Proposition 4, reducing the valuation of a seller type that is not the highest introduces slack in the binding incentive constraint for that type if the same allocation is to be implemented; this decreases the information rent needed to compensate that type relative to the next higher type. However, this potentially positive welfare effect is offset by the direct loss created by the damage as well as by the increased transfers required to meet the seller's damage participation constraint. In particular, the damage participation constraint for the seller with the highest valuation is more restrictive than the post-damage individual rationality constraint for this type; a higher expected transfer is required for this type to implement the same allocation. The proof of Proposition 5 shows that under damage at the top, the additional transfer imposed by the damage participation constraint for the seller type with the highest valuation is greater than the sum of transfers saved on lower types as a result of the diminished information rents dispensed to these types after the damage.

## 7. RESULTS FOR CONTINUOUS TYPES

In this section, we present versions of our results for the original Myerson-Satterthwaite model, in which buyer and seller valuations have continuous distributions. Assume that the valuations of the buyer and the seller have cumulative distribution functions  $F_b$  and  $F_s$  with continuous densities  $f_b = F'_b$  and  $f_s = F'_s$  that are positive at every point in their respective supports  $[b, \bar{b}]$  and  $[s, \bar{s}]$ . Recall that the virtual values of the buyer of type  $b$  and the seller of type  $s$  are defined by

$$\psi_b^{F_b}(b) = b - \frac{1 - F_b(b)}{f_b(b)} \text{ and } \psi_s^{F_s}(s) = s + \frac{F_s(s)}{f_s(s)}, \text{ respectively.}$$

A buyer distribution  $F_b$  or a seller distribution  $F_s$  is said to be *regular* if its respective virtual value function  $\psi_b^{F_b}$  or  $\psi_s^{F_s}$  is strictly increasing.

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<sup>8</sup>If the mechanism designer could damage the good without the seller's consent—i.e., has to abide by the individual rationality constraint with respect to the seller's reduced valuation, but not by the more restrictive damage participation constraint—then a modification of the example above demonstrates that damage at the top may increase welfare under the optimal mechanism. This is possible even when the mechanism designer is subject to an ex ante damage participation constraint requiring that the seller's expected payoff in the post-damage mechanism is not smaller than the seller's expected value for the undamaged good. See the Appendix for a discussion.

In the continuous-type setting, a direct mechanism  $(x_b, x_s, t_b, t_s)$  that permits withholding is described by allocation functions  $x_b, x_s : [\underline{b}, \bar{b}] \times [\underline{s}, \bar{s}] \rightarrow [0, 1]$  and transfer functions  $t_b, t_s : [\underline{b}, \bar{b}] \times [\underline{s}, \bar{s}] \rightarrow \mathbb{R}$  that are Riemann integrable. Constraints for an implementable mechanism are specified analogously to the discrete-type setting (with integrals replacing summations).

The first result for the continuous-type setting establishes the counterpart of Proposition 1—money burning is never optimal. The proof relies on a perturbation similar to the one used to prove Proposition 1.

**Proposition 6.** *Suppose that  $(x_b, x_s, t_b, t_s)$  is an optimal implementable mechanism for a pair of value distributions  $(F_b, F_s)$ . Then,*

$$\int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(b, s) - t_s(b, s)) f_b(b) f_s(s) db ds = 0.$$

Given this result, in solving for optimal mechanisms we can now assume that  $t_b(b, s) = t_s(b, s)$  for all pairs of types  $(b, s)$ . We simplify notation for a mechanism to  $(x_b, x_s, t)$  with the understanding that  $t_b(b, s) = t_s(b, s) = t(b, s)$ .

We next prove the analog of Proposition 2—withholding the good with positive probability is not optimal—under the assumption that buyer and seller value distributions are regular. The proof shows that the optimal mechanism characterized by Myerson and Satterthwaite in the case of regular value distributions continues to be optimal when the designer is given the extra flexibility to withhold the good.

**Proposition 7.** *Under any optimal implementable mechanism  $(x_b, x_s, t)$  for a pair of regular value distributions  $(F_b, F_s)$ , the set of pairs of types  $(b, s)$  for which  $x_b(b, s) + x_s(b, s) \neq 1$  has probability zero under the product distribution  $F_b \times F_s$ .*

In light of Proposition 7, the rest of this section assumes that  $x_b(b, s) + x_s(b, s) = 1$  for all pairs  $(b, s)$ . A mechanism is now simply denoted by  $(x, t)$ , with allocation functions given by  $x_b(b, s) = x(b, s)$  and  $x_s(b, s) = 1 - x(b, s)$ . This is the universe of mechanisms considered by Myerson and Satterthwaite from the outset of their analysis. However, we do not maintain the hypothesis of Proposition 7 that value distributions are regular.

The next result extends Proposition 3 to the continuous-type setting under the additional assumption that the damage for each buyer type results in a reduction in corresponding virtual values. Specifically, we interpret damaging the good for the buyer as a first-order stochastic dominance shift from  $F_b$  to  $G_b$  whereby the value of the buyer of type  $b$  is reduced from  $b$  to  $\alpha(b) := G_b^{-1} \circ F_b(b)$ , and require that the virtual value of type  $b$  under the original distribution  $F_b$  is greater than or equal to the virtual value of the corresponding type  $\alpha(b)$  under the post-damage distribution  $G_b$ :  $\psi_b^{F_b}(b) \geq \psi_b^{G_b}(\alpha(b))$ . Since  $G_b(\alpha(b)) = F_b(b)$  and  $g_b(\alpha(b))\alpha'(b) = f_b(b)$ , this sufficient condition is equivalent to

$$(4) \quad b - \alpha(b) \geq (1 - \alpha'(b)) \frac{1 - F_b(b)}{f_b(b)}.$$

The inequality above can be rewritten as

$$\frac{f_b(b)}{1 - F_b(b)} \geq \frac{1 - \alpha'(b)}{b - \alpha(b)} \iff (\log [(b - \alpha(b))(1 - F_b(b))])' \leq 0,$$

which is equivalent to  $(b - \alpha(b))(1 - F_b(b))$  being a decreasing function of  $b$ .

The proof combines Myerson and Satterthwaite's feasibility condition for implementing monotonic allocations with the intuition behind Proposition 3 that an implementable mechanism under  $(G_b, F_s)$  can be modified to obtain an implementable mechanism under  $(F_b, F_s)$  by redirecting trade with type  $\alpha(b)$  under  $G_b$  to type  $b$  under  $F_b$  in a way that does not affect trading probabilities for individual seller types.

**Proposition 8.** *Let  $F_b$  and  $G_b$  be distributions of buyer values such that  $F_b$  first-order stochastically dominates  $G_b$ , and denote  $\alpha = G_b^{-1} \circ F_b$ . Suppose that  $\psi_b^{F_b}(b) \geq \psi_b^{G_b}(\alpha(b))$  for all  $b$  in the support of  $F_b$ . Then, for every seller value distribution  $F_s$ , the total welfare under an optimal implementable mechanism for the pair of value distributions  $(G_b, F_s)$  does not exceed the one for  $(F_b, F_s)$ .*

The final result for the continuous-type setting extends Proposition 5 under a different version of the hypothesis that no seller type is hurt by the damage more than the type with the highest valuation. The new hypothesis presumes that damage for seller type  $s$  whose value is reduced to  $\alpha(s)$  via a first-order stochastic dominance shift from  $F_s$  to  $G_s$  is reflected by the difference in virtual values of type  $s$  under the distribution  $F_s$  and the corresponding type  $\alpha(s)$  under the distribution  $G_s$ . The hypothesis requires that the reduction in virtual values  $\psi_s^{F_s}(s) - \psi_s^{G_s}(\alpha(s))$  for every type  $s$  should not exceed the reduction in actual values  $\bar{s} - \alpha(\bar{s})$  for the highest type  $\bar{s}$ .<sup>9</sup> As in the discrete-type setting, post-damage mechanisms must obey the seller's damage participation constraint, which requires that for every  $s$  in the support of  $F_s$ , the seller type with reduced valuation  $\alpha(s)$  receives an expected utility of at least  $s$  from participating in the mechanism after the damage.

**Proposition 9.** *Let  $F_s$  and  $G_s$  be distributions of seller values such that  $F_s$  first-order stochastically dominates  $G_s$ , and denote  $\alpha = G_s^{-1} \circ F_s$ . Suppose that  $\psi_s^{F_s}(s) - \psi_s^{G_s}(\alpha(s)) \leq \bar{s} - \alpha(\bar{s})$  for all  $s$  in the support of  $F_s$ , where  $\bar{s}$  denotes the maximum of the support of  $F_s$ . Then, there is no implementable mechanism for the pair of value distributions  $(F_b, G_s)$  that obeys the seller's damage participation constraint for the reduction in values from  $F_s$  to  $G_s$  and achieves greater total welfare than the optimal mechanism for  $(F_b, F_s)$ .*

Propositions 7 through 9 invoke hypotheses related to virtual values that were not needed for their discrete-type counterparts. These hypotheses would appear as restrictive in the context of discrete value distributions as they are for continuous ones;<sup>10</sup> in this sense, our results for the discrete-type setting are stronger. Whether the results for continuous types continue to hold if we drop the hypotheses pertaining to virtual values remains a question for future

<sup>9</sup>For  $s = \bar{s}$ , the hypothesis requires that  $\alpha'(\bar{s}) \leq 1$ .

<sup>10</sup>The appropriate definition of the virtual value of buyer type  $b_i$  drawn from discrete distribution  $p_b$  is given by  $b_i - (b_{i+1} - b_i)(p_b(b_{i+1}) + \dots + p_b(b_m))/p_b(b_i)$  (e.g., Lovejoy 2006); there is an analogous adaptation for the seller's virtual values.

research. One approach that might be useful in extending our results to all continuous distributions would be to develop convergent upper and lower bounds on total welfare under the optimal mechanism for any pair of continuous distributions via discrete grid approximations, and derive Propositions 7, 8 and 9 from Propositions 2, 3 and 5, respectively.<sup>11</sup> The underlying convergence argument would entail showing that the value of the linear program associated with the optimal mechanism for discrete value distributions varies continuously with the distribution (within the class of discrete distributions employed to derive upper and lower bounds). This is a separate hurdle since the optimal value of a linear program is not always a continuous function of coefficients in the constraints (see Gale 1960 (p. 95), Williams 1963 and Wets 1985).

## 8. ALLOCATION MECHANISMS WITHOUT PROPERTY RIGHTS

We extend our analysis to the model of social choice rules formulated by Wolitzky (2016). A set  $N$  of  $n \geq 1$  agents face a set of alternatives  $X \subset [0, \infty)^n$ . The type of each agent  $i$  is drawn independently from a set of nonnegative numbers  $V_i$ , which is either finite or a closed interval. Agents have a common prior, and agent  $i$  is privately informed about his type  $v_i$ . The set of type profiles is denoted by  $V = \prod_{i \in N} V_i$ . If alternative  $x \in X$  is implemented, then an agent  $i$  of type  $v_i$  who makes a payment  $t_i$  gets payoff  $x_i v_i - t_i$ . Special instances of this framework include allocation problems for multiple units of an indivisible good for which agents have unit demand (in this case,  $x_i$  is the probability that agent  $i$  receives a unit) and allocation problems for a divisible good (in this case,  $x_i$  is the quantity agent  $i$  receives).<sup>12</sup>

In this environment, a direct mechanism specifies an allocation function  $x : V \rightarrow X$  and a transfer function  $t : V \rightarrow \mathbb{R}^n$ . The *IC*, *IR* and *FT* constraints and total welfare for Bayesian implementable mechanisms are defined analogously to those for bilateral trade (the *FX* constraint is encapsulated in the requirement that the range of  $x$  is  $X$ ).<sup>13</sup>

We assume that the set of alternatives  $X$  is convex and compact. Convexity captures the ability of the mechanism designer to randomize over alternatives, and compactness ensures the existence of optimal implementable mechanisms.

The equivalent of Propositions 1 and 6 in this new environment is that for every optimal implementable mechanism  $(x, t)$ , all *FT* constraints hold with equality, that is,  $\sum_{i \in N} t_i(v) = 0$  for all  $v \in V$  (with probability 1). Adapting the proofs for both the discrete- and the continuous-type settings requires primarily notational adjustments.

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<sup>11</sup>Madarasz and Prat (2017) carry on a related exercise in a screening problem with a single agent, and construct a mechanism in which the agent's optimal decisions may respond discontinuously to approximations of his type, but affect the principal's profit continuously. In our setting, perturbations in the mechanism affect the optimal reports of both the buyer and the seller, and the challenge is to maintain the continuity of the allocation resulting from this interaction.

<sup>12</sup>This framework does *not* formally nest our bilateral trade setting because it rules out property rights. Wolitzky applies it to bilateral trade by assuming that the seller does not hold the good, but instead produces it ex post at a privately known cost only when the mechanism prescribes that trade take place.

<sup>13</sup>In contrast to the Bayesian approach we maintain in this analysis, Wolitzky studies the case of agents with maxmin preferences.

Proposition 3 also admits a straightforward extension to the discrete-type version of this environment: a reduction in any agent’s valuation in terms of first-order stochastic dominance cannot enhance welfare under the optimal implementable mechanism. As in the case of bilateral trade, the main step in proving this claim relies on the assumed convexity of the set  $X$  of feasible alternatives. Similarly, Proposition 8 extends to the continuous-type version of the current environment under the same hypothesis regarding the effect of damage on the agent’s virtual values.<sup>14</sup>

To draw parallels with Propositions 2 and 7 in this environment, consider an instance in which the set of alternatives  $X$  represents a multi-unit allocation problem in which agents have unit demand. In this setting, for both discrete and continuous value distributions that are *regular, ex post efficient* allocations are implementable—hence, withholding goods is categorically inefficient—under the maintained assumptions of Bayesian incentive compatibility, ex post budget balance and interim individual rationality. Drexl and Kleiner (2015) establish this result for an allocation problem with two agents and a single good in which agents have regular continuous value distributions, but their argument generalizes to problems with larger numbers of agents and goods. Therefore, the results of Guo and Conitzer (2008, 2009), Moulin (2009) and de Clippel et al. (2014) regarding the necessity of burning money and the optimality of withholding goods in this setting hinge on their imposition of the stronger solution concept of dominant strategy incentive compatibility.

## 9. CONCLUSION

In this paper, we investigated the optimality of withholding and damaging resources in the context of Bayesian implementation for bilateral trade and allocation problems with independent private values. In the bilateral trade setting with discrete value distributions, we established that neither burning money nor withholding the good can enhance welfare. We found that the welfare effects of damaging the good are driven by the asymmetry in property rights between the two traders: damage for the buyer cannot improve welfare, while damage for the seller can. Nevertheless, damaging the good for the seller may enhance welfare only if it hurts a seller type with a low valuation more than the type with the highest valuation. We hope that the perturbation techniques we developed in this analysis will prove useful in other mechanism design problems with discrete types.

Extending our results to the bilateral trade setting with continuous value distributions required introducing some hypotheses about virtual values. It is an open question whether these hypotheses are necessary for the continuous-type results.

We also expanded our analysis to allocation problems without property rights. In this environment, we showed that damaging the good for any agent does not increase welfare, mirroring the result from the bilateral trade model that concerns damage for the buyer. We argued that money burning is never optimal, while withholding goods is not optimal if agents have regular value distributions. The divergence of these conclusions from existing research

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<sup>14</sup>Propositions 4, 5 and 9 do not have natural equivalents in this environment because the seller’s ownership of the good is central to our analysis of damage for the seller. All agents act as buyers here.

can be attributed to our focus on Bayesian implementation in contrast to the more demanding criterion of dominant strategy implementation (coupled with the desideratum of no ex post budget deficit) commonly imposed in the literature.

## APPENDIX: PROOFS

*Proof of Proposition 1.* Let  $(x_b, x_s, t_b, t_s)$  be an optimal implementable mechanism. We prove by contradiction that  $t_b(b_i, s_j) = t_s(b_i, s_j)$  for all pairs of types  $(b_i, s_j)$ . Suppose that there exists a pair  $(b_i, s_j)$  with  $\delta := t_b(b_i, s_j) - t_s(b_i, s_j) > 0$ . Consider a perturbation  $(x_b, x_s, t'_b, t'_s)$  of the mechanism  $(x_b, x_s, t_b, t_s)$  specifying that the transfer to seller type  $s_j$  is increased to match the transfer from  $b_i$  when the buyer reports type  $b_i$  and that the transfers  $s_j$  receives from all buyer types are uniformly reduced and credited to the buyer, so that  $s_j$  has the same expected utility following the perturbation. The perturbed mechanism  $(x_b, x_s, t'_b, t'_s)$  is identical to  $(x_b, x_s, t_b, t_s)$  except for the following profiles of reports:

$$\begin{aligned} t'_s(b_i, s_j) &= t_s(b_i, s_j) + \delta - p_b(b_i)\delta \\ t'_s(b_k, s_j) &= t_s(b_k, s_j) - p_b(b_i)\delta, \forall k = \overline{1, m}, k \neq i \\ t'_b(b_k, s_j) &= t_b(b_k, s_j) - p_b(b_i)\delta, \forall k = \overline{1, m}. \end{aligned}$$

The definition is so that  $(\bar{x}_b, \bar{x}_s, \bar{t}'_b, \bar{t}'_s)$  is identical to  $(\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s)$  at all values with the following exception:

$$\bar{t}'_b(b_k) = \bar{t}_b(b_k) - p_b(b_i)p_s(s_j)\delta, \forall k = \overline{1, m}.$$

We can easily compare expected outcomes under the mechanism  $(x_b, x_s, t'_b, t'_s)$  to those under  $(x_b, x_s, t_b, t_s)$ . Each buyer type receives the good with the same probability and expects a uniform decrease in transfers. Each seller type obtains the good with the same probability and receives the same expected transfer. Therefore,  $(x_b, x_s, t'_b, t'_s)$  satisfies all buyer and seller rationality and incentive constraints because  $(x_b, x_s, t_b, t_s)$  does. We conclude that the mechanism  $(x_b, x_s, t'_b, t'_s)$  is implementable and yields an increase in welfare of  $p_b(b_i)p_s(s_j)\delta > 0$  over the mechanism  $(x_b, x_s, t_b, t_s)$ , a contradiction with the optimality of the latter.  $\square$

*Proof of Lemma 1.* We need to show that the set of constraints enumerated in the statement imply all the other ones.  $IC_{b_i \rightarrow b_{i+1}}$  and  $IC_{b_{i+1} \rightarrow b_i}$  lead to

$$(5) \quad b_i(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)) \leq \bar{t}_b(b_{i+1}) - \bar{t}_b(b_i) \leq b_{i+1}(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)), \forall i = \overline{1, m-1}.$$

In particular,  $b_i(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)) \leq b_{i+1}(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i))$  and  $b_{i+1} \geq b_i$  imply that

$$(6) \quad \bar{x}_b(b_{i+1}) \geq \bar{x}_b(b_i), \forall i = \overline{1, m-1}.$$

If  $IC_{b_{i+1} \rightarrow b_i}$  holds with equality, then the second inequality in (5) holds with equality. If additionally  $\bar{x}_b(b_{i+1}) > \bar{x}_b(b_i)$ , then the first expression in (5) is strictly smaller than the last expression in (5), so the first inequality must be strict, which means that  $IC_{b_i \rightarrow b_{i+1}}$  is satisfied with strict inequality.

If  $i < k \leq m$ , then (5) and (6) imply that

$$\begin{aligned} b_i(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)) &\leq b_g(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)) \leq \bar{t}_b(b_{g+1}) - \bar{t}_b(b_g) \\ &\leq b_{g+1}(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)) \leq b_k(\bar{x}_b(b_{g+1}) - \bar{x}_b(b_g)), \forall g = \overline{i, k-1}. \end{aligned}$$

Adding up these inequalities, we obtain

$$b_i(\bar{x}_b(b_k) - \bar{x}_b(b_i)) \leq \bar{t}_b(b_k) - \bar{t}_b(b_i) \leq b_k(\bar{x}_b(b_k) - \bar{x}_b(b_i)),$$

which is equivalent to  $IC_{b_i \rightarrow b_k}$  and  $IC_{b_k \rightarrow b_i}$ .

Also,  $IC_{b_k \rightarrow b_1}$  and  $IR_{b_1}$  imply that

$$b_k \bar{x}_b(b_k) - \bar{t}_b(b_k) \geq b_k \bar{x}_b(b_1) - \bar{t}_b(b_1) \geq b_1 \bar{x}_b(b_1) - \bar{t}_b(b_1) \geq 0,$$

verifying  $IR_{b_k}$  for  $k = \overline{2, m}$ .

The claims regarding the seller are checked similarly.  $\square$

*Proof of Lemma 2.* Among the monetary transfer functions  $t$  that implement  $(x_b, x_s)$ , there exists one,  $t'$ , that maximizes the expected payoff of seller type  $s_n$ . We show that  $t'$  has the desired properties. Suppose that  $IC_{b_{i+1} \rightarrow b_i}$  holds with strict inequality under  $(x_b, x_s, t')$ . For  $\varepsilon > 0$  sufficiently small, the transfer function  $t''$  defined by

$$\begin{aligned} t''(b_k, s_j) &= t'(b_k, s_j) \text{ for } k \leq i, j = \overline{1, n} \\ t''(b_k, s_j) &= t'(b_k, s_j) + \varepsilon \text{ for } k \geq i+1, j = \overline{1, n} \end{aligned}$$

satisfies the sufficient individual rationality and incentive constraints from Lemma 1, and increases the expected payoff of  $s_n$  by  $\varepsilon \sum_{k=i+1}^n p_b(b_k) > 0$ , a contradiction with the definition of  $t'$ . The conclusion regarding  $IR_{b_1}$  follows from perturbing the transfers as in the formulae above with  $i = 0$ .

We obtain a similar contradiction if  $(x_b, x_s, t')$  satisfies  $IC_{s_j \rightarrow s_{j+1}}$  with strict inequality by perturbing the transfer function as follows:

$$\begin{aligned} t''(b_i, s_l) &= t'(b_i, s_l) - \frac{\varepsilon}{\sum_{l'=1}^j p_s(s_{l'})} \text{ for } l \leq j, i = \overline{1, m} \\ t''(b_i, s_l) &= t'(b_k, s_j) + \frac{\varepsilon}{\sum_{l'=j+1}^n p_s(s_{l'})} \text{ for } l \geq j+1, i = \overline{1, m}. \end{aligned}$$

This perturbation keeps buyer's incentives in place because the expected transfer of each buyer type under  $t''$  is the same as under  $t$ .  $\square$

*Proof of Proposition 2.* We prove the second part of the statement proceeding by contradiction. Suppose that  $s_1 > 0$ , and let  $(x_b, x_s, t)$  be an optimal implementable mechanism under which the good is withheld with positive probability for some buyer-seller value profiles. We can assume without loss of generality that  $t$  is selected such that  $(x_b, x_s, t)$  satisfies all the constraints in Lemma 2 with equality. Let  $(b_i, s_j)$  be a pair of buyer and seller types where the good is withheld with positive probability, i.e.,  $x_b(b_i, s_j) + x_s(b_i, s_j) < 1$ . Let  $j'$  be the highest seller type such that  $\bar{x}_s(s_j) = \bar{x}_s(s_{j'})$ . By Lemma 1, since  $(x_b, x_s, t)$  satisfies  $IC_{s_{j'} \rightarrow s_{j'+1}}$  with equality and  $\bar{x}_s(s_{j'}) < \bar{x}_s(s_{j'+1})$ , it must satisfy  $IC_{s_{j'+1} \rightarrow s_{j'}}$  with strict inequality.

**Step 1.** We show that  $j \neq j'$  and  $j \neq n$ .

For a proof by contradiction, suppose that  $j = j'$  or  $j = n$ . Consider the mechanism  $(x'_b, x'_s, t')$ , which perturbs  $(x_b, x_s, t)$  at the following values:

$$\begin{aligned} x'_s(b_i, s_j) &= x_s(b_i, s_j) + \varepsilon \\ t'(b_k, s_j) &= t(b_k, s_j) - s_j p_b(b_i) \varepsilon, \forall k = \overline{1, m}. \end{aligned}$$

For small  $\varepsilon > 0$ ,  $x'_b(b_i, s_j) + x'_s(b_i, s_j) \leq 1$ . The definition is so that  $(\bar{x}'_b, \bar{x}'_s, \bar{t}'_b, \bar{t}'_s)$  take the same values as  $(\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s)$  with the following exceptions:

$$\begin{aligned} \bar{x}'_s(s_j) &= \bar{x}_s(s_j) + p_b(b_i) \varepsilon \\ \bar{t}'_s(s_j) &= \bar{t}_s(s_j) - s_j p_b(b_i) \varepsilon \\ \bar{t}'_b(b_k) &= \bar{t}_b(b_k) - s_j p_b(b_i) p_s(s_j) \varepsilon, \forall k = \overline{1, m}. \end{aligned}$$

We compare expected outcomes under the mechanism  $(x'_b, x'_s, t')$  to those under  $(x_b, x_s, t)$ . For each type report, the buyer receives the good with the same probability, and the expected transfer is uniformly reduced by a constant. Hence,  $(x'_b, x'_s, t')$  satisfies all buyer rationality and incentive constraints because  $(x_b, x_s, t)$  does.

The mechanism  $(x'_b, x'_s, t')$  satisfies all seller constraints that do not involve type  $s_j$  because  $(\bar{x}'_s, \bar{t}'_s)$  and  $(\bar{x}_s, \bar{t}_s)$  coincide for those types. Moreover, seller type  $s_j$  expects the same utility from reporting any given type under  $(x'_b, x'_s, t')$  and  $(x_b, x_s, t)$ , so  $(x'_b, x'_s, t')$  also satisfies the constraints involving the rationality and incentives of type  $s_j$ . We are left to check that  $(x'_b, x'_s, t')$  satisfies the constraints  $IC_{s_{j-1} \rightarrow s_j}$  and  $IC_{s_{j+1} \rightarrow s_j}$  from Lemma 1.

$IC_{s_{j-1} \rightarrow s_j}$  for  $(x'_b, x'_s, t')$  is equivalent to

$$\begin{aligned} \bar{t}_s(s_{j-1}) - (1 - \bar{x}_s(s_{j-1}))s_{j-1} &\geq (\bar{t}_s(s_j) - s_j p_b(b_i) \varepsilon) - (1 - (\bar{x}_s(s_j) + p_b(b_i) \varepsilon))s_{j-1} \\ \iff \bar{t}_s(s_{j-1}) - (1 - \bar{x}_s(s_{j-1}))s_{j-1} &\geq \bar{t}_s(s_j) - (1 - \bar{x}_s(s_j))s_{j-1} - (s_j - s_{j-1})p_b(b_i) \varepsilon, \end{aligned}$$

which follows from  $s_j > s_{j-1}$  and the fact that  $(x_b, x_s, t)$  satisfies  $IC_{s_{j-1} \rightarrow s_j}$ .

For small  $\varepsilon > 0$ ,  $(x'_b, x'_s, t')$  satisfies  $IC_{s_{j+1} \rightarrow s_j}$  by continuity since  $(x'_b, x'_s, t')$  satisfies  $IC_{s_{j+1} \rightarrow s_j}$  with strict inequality (for  $j = j'$ , this was argued immediately after the definition of  $j'$ ; for  $j = n$ , the argument is unnecessary).

Therefore, for small  $\varepsilon > 0$ , the mechanism  $(x'_b, x'_s, t')$  is implementable and yields an increase in welfare of  $s_j p_b(b_i) p_s(s_j) \varepsilon > 0$  over the mechanism  $(x_b, x_s, t)$ , a contradiction with the optimality of  $(x_b, x_s, t)$ .

**Step 2.** It must be that  $x_b(b_{i'}, s_{j'}) + x_s(b_{i'}, s_{j'}) = 1$  for all  $i' = \overline{1, m}$ .

This follows by replacing  $i$  with  $i'$  everywhere in the argument from Step 1.

**Step 3.** We have that  $x_b(b_i, s_{j'}) = 0$  and  $x_s(b_i, s_{j'}) = 1$ .

Steps 1 and 2 show that  $j < j'$  and  $x_b(b_i, s_{j'}) + x_s(b_i, s_{j'}) = 1$ . Suppose, for a contradiction, that  $x_b(b_i, s_{j'}) > 0$ .

Construct the mechanism  $(x'_b, x'_s, t')$  identical to  $(x_b, x_s, t)$  on  $V_b \times V_s$  with the following exceptions:

$$\begin{aligned} x'_b(b_i, s_j) &= x_b(b_i, s_j) + \varepsilon \\ x'_b(b_i, s_{j'}) &= x_b(b_i, s_{j'}) - \frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon \\ x'_s(b_i, s_{j'}) &= x_s(b_i, s_{j'}) + \frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon \\ t'(b_k, s_{j'}) &= t(b_k, s_{j'}) - s_{j'}p_b(b_i)\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon, \forall k = \overline{1, m}. \end{aligned}$$

For small  $\varepsilon > 0$ , we have  $x'_b(b_i, s_j) + x'_s(b_i, s_j) \leq 1$  and  $x'_b(b_i, s_{j'}) \geq 0$ .

Arguments similar to those used in Step 1 show that the mechanism  $(x'_b, x'_s, t')$  is implementable for small  $\varepsilon > 0$ . We reach the contradiction that  $(x'_b, x'_s, t')$  generates

$$s_{j'}p_s(s_{j'})p_b(b_i)\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon = s_{j'}p_b(b_i)p_s(s_j)\varepsilon > 0$$

more surplus than  $(x_b, x_s, t)$ .

**Step 4.** We reach the final contradiction.

Steps 1 and 3 establish that  $j < j'$  and  $x_s(b_i, s_{j'}) = 1$ . In particular,  $x_s(b_i, s_j) \leq x_b(b_i, s_j) + x_s(b_i, s_j) < 1 = x_s(b_i, s_{j'})$ . Since  $x_s(b_i, s_j) < x_s(b_i, s_{j'})$ ,  $p_b(b_i) > 0$ , and  $\bar{x}_s(s_j) = \bar{x}_s(s_{j'})$ , there exist a buyer type  $b_{i'}$  such that  $x_s(b_{i'}, s_j) > x_s(b_{i'}, s_{j'})$ . It follows that  $x_s(b_{i'}, s_j) > 0$ ,  $x_b(b_{i'}, s_j) < 1$ , and  $x_s(b_{i'}, s_{j'}) < 1$ . By Step 2,  $x_b(b_{i'}, s_{j'}) + x_s(b_{i'}, s_{j'}) = 1$ , and hence  $x_b(b_{i'}, s_{j'}) > 0$ . We collect the relevant inequalities:

$$(7) \quad x_s(b_i, s_j) < 1, \quad x_s(b_{i'}, s_j) > 0, \quad x_b(b_{i'}, s_j) < 1, \quad x_b(b_{i'}, s_{j'}) > 0, \quad x_s(b_{i'}, s_{j'}) < 1.$$

Construct a mechanism  $(x'_b, x'_s, t')$  that perturbs  $(x_b, x_s, t)$  for the following types:

$$\begin{aligned} x'_s(b_i, s_j) &= x_s(b_i, s_j) + \varepsilon \\ x'_s(b_{i'}, s_j) &= x_s(b_{i'}, s_j) - \frac{p_b(b_i)}{p_b(b_{i'})}\varepsilon \quad \& \quad x'_b(b_{i'}, s_j) = x_b(b_{i'}, s_j) + \frac{p_b(b_i)}{p_b(b_{i'})}\varepsilon \\ x'_b(b_{i'}, s_{j'}) &= x_b(b_{i'}, s_{j'}) - \frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon \quad \& \quad x'_s(b_{i'}, s_{j'}) = x_s(b_{i'}, s_{j'}) + \frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon \\ t'(b_k, s_{j'}) &= t(b_k, s_{j'}) - s_{j'}p_b(b_{i'})\frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon = t(b_k, s_{j'}) - s_{j'}p_b(b_i)\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon, \forall k = \overline{1, m}. \end{aligned}$$

For small  $\varepsilon > 0$ , the set of inequalities (7) guarantees that all values of  $x'$  belong to the interval of probabilities  $[0, 1]$ . We can then argue as in Steps 1 and 3 that for small  $\varepsilon > 0$ , the mechanism  $(x'_b, x'_s, t')$  is implementable and improves the welfare of  $(x_b, x_s, t)$  by

$$s_{j'}p_b(b_{i'})p_s(s_{j'})\frac{p_b(b_i)}{p_b(b_{i'})}\frac{p_s(s_j)}{p_s(s_{j'})}\varepsilon = s_{j'}p_s(s_j)p_b(b_i)\varepsilon > 0,$$

a contradiction.

Steps 1 through 4 show that if  $(x_b, x_s, t)$  is an optimal implementable mechanism, then  $x_b(b_i, s_j) + x_s(b_i, s_j) = 1$ .

To establish the first part of the result, note that the set of optimal mechanisms  $(x, t)$  is a compact subset of an Euclidean space. It follows that there exists such a mechanism that minimizes the total probability of withholding. If this mechanism withholds the good with positive probability, the arguments above lead to a contradiction (and also that withholding under an optimal mechanism is possible only in the case  $s_1 = 0$  and withholding takes place when the seller reports type  $s_1$ ).  $\square$

*Proof of Proposition 3.* As discussed in the proof sketch, it is sufficient to establish the result for cases in which  $p_b(b_i) = p'_b(b_i) - \delta$  and  $p_b(b_{i+1}) = p'_b(b_{i+1}) + \delta$  with  $\delta > 0$ , and  $p'_b(b_k) = p_b(b_k)$  for  $k \neq i, i + 1$ .

Let  $(x', t')$  be a mechanism that maximizes welfare when traders' values are distributed according to  $(p'_b, p_s)$  and satisfies all constraints in Lemma 2 with equality (we use only the equality in the constraint  $IC_{b_{i+1} \rightarrow b_i}$  for this proof). Define the mechanism  $(x, t)$  to coincide with  $(x', t')$  for all type profiles with the following exceptions:

$$\begin{aligned} x(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})} x'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})} x'(b_{i+1}, s_j), \forall j = \overline{1, n} \\ t(b_{i+1}, s_j) &= \frac{\delta}{p_b(b_{i+1})} t'(b_i, s_j) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})} t'(b_{i+1}, s_j), \forall j = \overline{1, n}. \end{aligned}$$

Denote by  $\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s$  and  $\bar{x}'_b, \bar{x}'_s, \bar{t}'_b, \bar{t}'_s$  the probabilities of trade and expected transfers for each buyer and seller type under the mechanism  $(x, t)$  when values are distributed according to  $(p_b, p_s)$  and under the mechanism  $(x', t')$  when values are distributed according to  $(p'_b, p_s)$ , respectively. The allocation  $x$  is defined so that

$$p_b(b_i)x(b_i, s_j) + p_b(b_{i+1})x(b_{i+1}, s_j) = p'_b(b_i)x'(b_i, s_j) + p'_b(b_{i+1})x'(b_{i+1}, s_j),$$

which implies that  $\bar{x}_s(s_j) = \bar{x}'_s(s_j)$  for all  $j$ . Similarly,  $\bar{t}_s(s_j) = \bar{t}'_s(s_j)$  for all  $j$ . Moreover,  $(\bar{x}'_b, \bar{x}'_s, \bar{t}'_b, \bar{t}'_s)$  is identical to  $(\bar{x}_b, \bar{x}_s, \bar{t}_b, \bar{t}_s)$  at all values except  $b_{i+1}$ , for which we have

$$\begin{aligned} \bar{x}_b(b_{i+1}) &= \frac{\delta}{p_b(b_{i+1})} \bar{x}'_b(b_i) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})} \bar{x}'_b(b_{i+1}) \\ \bar{t}_b(b_{i+1}) &= \frac{\delta}{p_b(b_{i+1})} \bar{t}'_b(b_i) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})} \bar{t}'_b(b_{i+1}). \end{aligned}$$

We show that the mechanism  $(x, t)$  satisfies all the *IR* and *IC* constraints in Lemma 1 when traders' values are distributed according to  $(p_b, p_s)$ . Since  $\bar{x}_b, \bar{t}_b, \bar{x}_s, \bar{t}_s$  coincide with  $\bar{x}'_b, \bar{t}'_b, \bar{x}'_s, \bar{t}'_s$  at all values except for  $b_{i+1}$ , the only constraints we need to check are those that involve buyer type  $b_{i+1}$ . Neither type  $b_i$  nor type  $b_{i+2}$  have an incentive to mimic type  $b_{i+1}$  in the mechanism  $(x, t)$  since each of these types is assigned the same allocation probabilities and expected transfers under  $(x, t)$  and  $(x', t')$ , and prefers his outcome under  $(x', t')$  to any other outcome achievable under  $(x', t')$  as well as to the convex combination of outcomes for types  $b_i$  and  $b_{i+1}$  specified by  $(x(b_{i+1}, \cdot), t(b_{i+1}, \cdot))$ .

We are left to check that  $(x, t)$  satisfies  $IC_{b_{i+1} \rightarrow b_i}$  and  $IC_{b_{i+1} \rightarrow b_{i+2}}$ . The constraint  $IC_{b_{i+1} \rightarrow b_i}$  requires that

$$\frac{\delta}{p_b(b_{i+1})}(\bar{x}'_b(b_i)b_{i+1} - \bar{t}'_b(b_i)) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}(\bar{x}'_b(b_{i+1})b_{i+1} - \bar{t}'_b(b_{i+1})) \geq \bar{x}'_b(b_i)b_{i+1} - \bar{t}'_b(b_i),$$

which is equivalent to

$$x'_b(b_{i+1})b_{i+1} - \bar{t}'_b(b_{i+1}) \geq \bar{x}'_b(b_i)b_{i+1} - \bar{t}'_b(b_i).$$

The latter inequality is a consequence of  $IC_{b_{i+1} \rightarrow b_i}$  under the mechanism  $(x', t')$ .

To check that  $(x, t)$  satisfies  $IC_{b_{i+1} \rightarrow b_{i+2}}$ , we need to show that

$$\frac{\delta}{p_b(b_{i+1})}(\bar{x}'_b(b_i)b_{i+1} - \bar{t}'_b(b_i)) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}(\bar{x}'_b(b_{i+1})b_{i+1} - \bar{t}'_b(b_{i+1})) \geq \bar{x}'_b(b_{i+2})b_{i+1} - \bar{t}'_b(b_{i+2}).$$

As  $(x', t')$  is assumed to satisfy  $IC_{b_{i+1} \rightarrow b_i}$  with equality, we have  $\bar{x}'_b(b_i)b_{i+1} - \bar{t}'_b(b_i) = \bar{x}'_b(b_{i+1})b_{i+1} - \bar{t}'_b(b_{i+1})$ , so the inequality above follows from  $IC_{b_{i+1} \rightarrow b_{i+2}}$  under  $(x', t')$ .

Since  $p_b, p_s, \bar{x}_b, \bar{t}_b, \bar{x}_s, \bar{t}_s$  coincide with  $p'_b, p_s, \bar{x}'_b, \bar{t}'_b, \bar{x}'_s, \bar{t}'_s$  at all values except for  $b_i$  and  $b_{i+1}$ , the difference in the welfare achieved by  $(x, t)$  with distributions  $(p_b, p_s)$  and  $(x', t')$  with distributions  $(p'_b, p_s)$  is given by

$$\begin{aligned} & p_b(b_i)\bar{x}_b(b_i)b_i + p_b(b_{i+1})\bar{x}_b(b_{i+1})b_{i+1} - p'_b(b_i)\bar{x}'_b(b_i)b_i - p'_b(b_{i+1})\bar{x}'_b(b_{i+1})b_{i+1} \\ &= p_b(b_i)\bar{x}'_b(b_i)b_i + p_b(b_{i+1}) \left( \frac{\delta}{p_b(b_{i+1})}\bar{x}'_b(b_i) + \frac{p_b(b_{i+1}) - \delta}{p_b(b_{i+1})}\bar{x}'_b(b_{i+1}) \right) b_{i+1} \\ & \quad - (p_b(b_i) + \delta)\bar{x}'_b(b_i)b_i - (p_b(b_{i+1}) - \delta)\bar{x}'_b(b_{i+1})b_{i+1} \\ &= \delta\bar{x}'_b(b_i)(b_{i+1} - b_i), \end{aligned}$$

which is non-negative. Therefore, the optimal mechanism for  $(p_b, p_s)$  yields at least the same amount of welfare as the optimal mechanism for  $(p'_b, p_s)$ .  $\square$

*Proof of Proposition 4.* Lemma 2 implies the existence of a transfer function  $t$  that implements the allocation  $x$  for the original pair of value distributions  $(p_b, p_s)$  such that  $(x, t)$  satisfies  $IC_{b_{i+1} \rightarrow b_i}$  with equality. Fix  $\varepsilon > 0$  and let  $s'_l = s_l - \varepsilon$  for  $l = \overline{1, j}$ . Define

$$(8) \quad \varepsilon' = \varepsilon \frac{(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) \sum_{l'=1}^j p_s(s_{l'})}{p_s(s_1)(b_{i+1} - b_i) \sum_{k'=i+1}^m p_b(b_{k'})}.$$

We seek to implement an allocation  $x'$  when the values of seller types  $l = \overline{1, j}$  are reduced from  $s_l$  to  $s'_l$  that differs from  $x$  (for corresponding seller types) only in that

$$x'(b_i, s'_1) = x(b_i, s_1) + \varepsilon'.$$

For this purpose, we perturb the transfer function  $t$  for the following pairs of types:

$$\begin{aligned} t'(b_i, s'_1) &= t(b_i, s_1) + \varepsilon' b_i \\ t'(b_k, s'_1) &= t(b_k, s_1) - \varepsilon'(b_{i+1} - b_i) \frac{p_s(s_1)}{\sum_{l'=1}^j p_s(s_{l'})} - \varepsilon'(b_i - s'_1) \frac{p_b(b_i)}{\sum_{k'=i+1}^m p_b(b_{k'})}, \forall k = \overline{i+1, m} \\ t'(b_k, s'_l) &= t(b_k, s_l) - \varepsilon'(b_{i+1} - b_i) \frac{p_s(s_1)}{\sum_{l'=1}^j p_s(s_{l'})}, \forall k = \overline{i+1, m}, l = \overline{2, j}. \end{aligned}$$

Transfers are specified so that buyer type  $b_i$  pays exactly his gain from the additional probability of receiving the good from seller type  $s_1$ . Similarly, we have that  $\bar{t}'_s(s'_1) - \bar{t}'_s(s'_2) = \bar{t}_s(s_1) - \bar{t}_s(s_2) + \varepsilon' p_b(b_i) s'_1$ , so that the new seller type  $s'_1$  is compensated relative to type  $s'_2$  for the utility lost by trading with buyer type  $b_i$  with the extra  $\varepsilon'$  probability.

To show that  $(x', t')$  is implementable following the value reduction, we verify that  $(x', t')$  satisfies the sufficient individual rationality and incentive constraints from Lemma 1 as well as the stricter participation constraints required for damaging the good for the seller.

Buyer expectations  $(\bar{x}'_b, \bar{t}'_b)$  under the new mechanism differ from the original ones in the following cases:

$$\begin{aligned}\bar{x}'_b(b_i) &= \bar{x}_b(b_i) + \varepsilon' p_s(s_1) \\ \bar{t}'_b(b_i) &= \bar{t}_b(b_i) + \varepsilon' p_s(s_1) b_i \\ \bar{t}'_b(b_k) &= \bar{t}_b(b_k) - \varepsilon' p_s(s_1)(b_{i+1} - b_i) - \varepsilon'(b_i - s'_1) \frac{p_b(b_i) p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})}, \forall k = \overline{i+1, m}.\end{aligned}$$

The mechanism  $(x', t')$  satisfies the incentive constraints  $IC_{b_k \rightarrow b_{k+1}}$  and  $IC_{b_{k+1} \rightarrow b_k}$  for  $k = \overline{i+1, m-1}$  since the involved buyer types receive the same allocation under  $x$  and  $x'$  and their expected payments are reduced by the same amount when shifting from  $t$  to  $t'$ . It is clear that  $(x', t')$  satisfies the constraints  $IC_{b_k \rightarrow b_{k+1}}$  and  $IC_{b_{k+1} \rightarrow b_k}$  for  $k = \overline{1, i-2}$  and  $IR_{b_1}$ .

We next check that  $(x', t')$  satisfies the incentive constraints  $IC_{b_{i-1} \rightarrow b_i}$ ,  $IC_{b_i \rightarrow b_{i-1}}$ ,  $IC_{b_i \rightarrow b_{i+1}}$  and  $IC_{b_{i+1} \rightarrow b_i}$ . The mechanism  $(x', t')$  satisfies  $IC_{b_i \rightarrow b_{i-1}}$  because buyer type  $b_i$  expects the same utility under  $(x, t)$  and  $(x', t')$  when he reports type  $b_i$ , and the same is true when he reports type  $b_{i-1}$ .  $IC_{b_{i-1} \rightarrow b_i}$  holds under  $(x', t')$  since for buyer type  $b_{i-1}$ , the marginal loss from misreporting his type to be  $b_i$  when we shift from  $(x, t)$  to  $(x', t')$  entails receiving the good with additional probability  $\varepsilon' p_s(s_1)$  for an expected benefit of  $\varepsilon' p_s(s_1) b_{i-1}$  at the greater cost of  $\varepsilon' p_s(s_1) b_i$ .

The buyer's incentives we are left to check for  $(x', t')$  are  $IC_{b_{i+1} \rightarrow b_i}$  and  $IC_{b_i \rightarrow b_{i+1}}$ . Since  $IC_{b_{i+1} \rightarrow b_i}$  holds with equality under  $(x, t)$ , we have that

$$(9) \quad \bar{x}_b(b_{i+1}) b_{i+1} - \bar{t}_b(b_{i+1}) = \bar{x}_b(b_i) b_{i+1} - \bar{t}_b(b_i).$$

To see that  $(x', t')$  satisfies  $IC_{b_{i+1} \rightarrow b_i}$ , we use (9) to infer that

$$\begin{aligned}\bar{x}'_b(b_{i+1}) b_{i+1} - \bar{t}'_b(b_{i+1}) &= \bar{x}_b(b_{i+1}) b_{i+1} - \bar{t}_b(b_{i+1}) + \varepsilon' p_s(s_1)(b_{i+1} - b_i) + \varepsilon'(b_i - s'_1) \frac{p_b(b_i) p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} \\ &= \bar{x}_b(b_i) b_{i+1} - \bar{t}_b(b_i) + \varepsilon' p_s(s_1)(b_{i+1} - b_i) + \varepsilon'(b_i - s'_1) \frac{p_b(b_i) p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} \\ &= \bar{x}'_b(b_i) b_{i+1} - \bar{t}'_b(b_i) + \varepsilon'(b_i - s'_1) \frac{p_b(b_i) p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} > \bar{x}'_b(b_i) b_{i+1} - \bar{t}'_b(b_i).\end{aligned}$$

The inequality follows from the hypothesis that  $b_i > s_1$ , which implies that  $b_i > s'_1$ .

Also using (9), we obtain that

$$\begin{aligned} \bar{x}'_b(b_i)b_i - \bar{t}'_b(b_i) &= \bar{x}_b(b_i)b_i - \bar{t}_b(b_i) = \bar{x}_b(b_{i+1})b_i - \bar{t}_b(b_{i+1}) + (b_{i+1} - b_i)(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)) \\ &= \bar{x}'_b(b_{i+1})b_i - \bar{t}'_b(b_{i+1}) - \varepsilon' p_s(s_1)(b_{i+1} - b_i) - \varepsilon'(b_i - s'_1) \frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} + (b_{i+1} - b_i)(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)). \end{aligned}$$

$IC_{b_i \rightarrow b_{i+1}}$  for  $(x', t')$  is thus equivalent to

$$\varepsilon' p_s(s_1)(b_{i+1} - b_i) + \varepsilon'(b_i - s'_1) \frac{p_b(b_i)p_s(s_1)}{\sum_{k'=i+1}^m p_b(b_{k'})} \leq (b_{i+1} - b_i)(\bar{x}_b(b_{i+1}) - \bar{x}_b(b_i)),$$

which holds for small  $\varepsilon > 0$  when  $\varepsilon'$  is given by (8) since  $\bar{x}_b(b_{i+1}) > \bar{x}_b(b_i)$  implies that the right-hand side of the inequality is positive.

We now turn to verifying the seller constraints for  $(x', t')$ . Seller expectations  $(\bar{x}'_s, \bar{t}'_s)$  under the perturbed mechanism differ from the corresponding ones in the original mechanism in the following instances:

$$\begin{aligned} \bar{x}'_s(s'_1) &= \bar{x}_s(s_1) + \varepsilon' p_b(b_i) \\ \bar{t}'_s(s'_1) &= \bar{t}_s(s_1) + \varepsilon' p_b(b_i) s'_1 - \varepsilon' p_s(s_1)(b_{i+1} - b_i) \frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})} \\ \bar{t}'_s(s'_l) &= \bar{t}_s(s_l) - \varepsilon' p_s(s_1)(b_{i+1} - b_i) \frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})}, \forall l = \overline{2, j}. \end{aligned}$$

We provide a proof for  $j \geq 2$  (the case  $j = 1$  involves similar arguments). Since  $\bar{x}_s(s_1) = \bar{x}_s(s_j)$ , it must be that  $\bar{x}_s(s_l) = \bar{x}_s(s_1)$  and  $\bar{t}_s(s_l) = \bar{t}_s(s_1)$  for  $l = \overline{2, j}$ . Hence  $\bar{x}'_s(s_l) = \bar{x}'_s(s_2)$  and  $\bar{t}'_s(s_l) = \bar{t}'_s(s_2)$  for  $l = \overline{2, j}$ . This implies that  $(x', t')$  satisfies  $IC_{s'_l \rightarrow s'_{l+1}}$  and  $IC_{s'_{l+1} \rightarrow s'_l}$  for  $l = \overline{2, j-1}$ . As  $(\bar{x}'_s, \bar{t}'_s)$  coincides with  $(\bar{x}_s, \bar{t}_s)$  for seller types  $l = \overline{j+1, n}$ , and  $(x, t)$  is implementable,  $(x', t')$  satisfies  $IC_{s_l \rightarrow s_{l+1}}$  and  $IC_{s_{l+1} \rightarrow s_l}$  for  $l = \overline{j+1, n-1}$  and  $IR_{s_n}$ . We are left to check the following incentive compatibility constraints from Lemma 1:  $IC_{s'_1 \rightarrow s'_2}$ ,  $IC_{s'_2 \rightarrow s'_1}$ ,  $IC_{s'_j \rightarrow s_{j+1}}$  and  $IC_{s_{j+1} \rightarrow s'_j}$ .

The constraints  $IC_{s'_1 \rightarrow s'_2}$  and  $IC_{s'_2 \rightarrow s'_1}$  for  $(x', t')$  are equivalent to

$$(\bar{x}'_s(s'_1) - \bar{x}'_s(s'_2))s'_1 \leq \bar{t}'_s(s'_1) - \bar{t}'_s(s'_2) \leq (\bar{x}'_s(s'_1) - \bar{x}'_s(s'_2))s'_2.$$

These inequalities follow from  $\bar{x}'_s(s'_1) - \bar{x}'_s(s'_2) = \varepsilon' p_b(b_i)$ ,  $\bar{t}'_s(s'_1) - \bar{t}'_s(s'_2) = \varepsilon' p_b(b_i) s'_1$  and  $s'_1 < s'_2$ .

To check  $IC_{s'_j \rightarrow s_{j+1}}$  for the mechanism  $(x', t')$ , note that  $IC_{s_j \rightarrow s_{j+1}}$  for  $(x, t)$  implies that

$$\begin{aligned} &\bar{t}'_s(s'_j) - \bar{t}'_s(s'_{j+1}) - (\bar{x}'_s(s'_j) - \bar{x}'_s(s'_{j+1}))s'_j \\ &= \bar{t}_s(s_j) - \bar{t}_s(s_{j+1}) - (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_j + \varepsilon(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) - \varepsilon' p_s(s_1)(b_{i+1} - b_i) \frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})} \\ &\geq \varepsilon(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1})) - \varepsilon' p_s(s_1)(b_{i+1} - b_i) \frac{\sum_{k'=i+1}^m p_b(b_{k'})}{\sum_{l'=1}^j p_s(s_{l'})} = 0, \end{aligned}$$

where the last equality follows from the definition of  $\varepsilon'$  in (8). The slack created in  $IC_{s_j \rightarrow s_{j+1}}$  by the  $\varepsilon$  value reduction for seller type  $j$  is used here to decrease the transfers from buyer types  $i+1$  through  $m$  to seller types 1 through  $j$ , which makes it possible to introduce the

slack in  $IC_{b_{i+1} \rightarrow b_i}$  necessary for increasing the probability of trade between seller type  $s_1$  and buyer type  $b_i$  at terms that would be otherwise attractive to buyer type  $b_{i+1}$ .

The mechanism  $(x', t')$  satisfies  $IC_{s_{j+1} \rightarrow s'_j}$  because  $(x, t)$  satisfies  $IC_{s_{j+1} \rightarrow s_j}$  and we have that  $\bar{x}'_s(s_{j+1}) = \bar{x}_s(s_{j+1})$ ,  $\bar{t}'_s(s_{j+1}) = \bar{t}_s(s_{j+1})$ ,  $\bar{x}'_s(s'_j) = \bar{x}_s(s_j)$  and  $\bar{t}'_s(s'_j) < \bar{t}_s(s_j)$ .

The arguments above establish that  $(x', t')$  is implementable for the pair of value distributions  $(p'_b, p_s)$ . We finally check that seller types  $l = \overline{1, j}$  have incentives to participate in the damage of the good. Since  $(x', t')$  satisfies  $IC_{s'_l \rightarrow s_{j+1}}$  and  $IR_{s_{j+1}}$  and  $\bar{x}'_s(s_{j+1}) = \bar{x}_s(s_{j+1}) > 0$ , we have that

$$\begin{aligned} \bar{t}'_s(s'_l) - \bar{x}'_s(s'_l)s'_l &\geq \bar{t}'_s(s_{j+1}) - \bar{x}'_s(s_{j+1})s'_l = \bar{t}'_s(s_{j+1}) - \bar{x}'_s(s_{j+1})s_{j+1} + \bar{x}_s(s_{j+1})(s_{j+1} - s'_l) \\ &\geq \bar{x}_s(s_{j+1})(s_{j+1} - s'_l) > \bar{x}_s(s_{j+1})(s_{j+1} - s_j). \end{aligned}$$

Then,

$$\bar{t}'_s(s'_l) + (1 - \bar{x}_s(s'_l))s'_l - s_l = \bar{t}'_s(s'_l) - \bar{x}_s(s'_l)s'_l + s'_l - s_l > \bar{x}_s(s_{j+1})(s_{j+1} - s_j) - \varepsilon.$$

The last term is positive if  $\varepsilon < \bar{x}_s(s_{j+1})(s_{j+1} - s_j)$ , and the proof is completed by noting that  $\bar{x}_s(s_{j+1})(s_{j+1} - s_j) > 0$ .  $\square$

*Proof of footnote 6.* Suppose that  $x(b_i, s_1) = 1$  and  $\bar{x}_s(s_1) < 1$ . Then, there exists a buyer type  $i'$  such that  $x(b_{i'}, s_1) < 1$ . Since  $\bar{x}_b(b_i) = \bar{x}_b(b_1) \leq \bar{x}_b(b_{i'})$  and  $x(b_{i'}, s_1) < 1 = x(b_i, s_1)$ , there must exist a seller type  $j'$  such that  $x(b_{i'}, s_{j'}) > x(b_i, s_{j'})$ . In particular, we have  $x(b_{i'}, s_{j'}) > 0$  and  $x(b_i, s_{j'}) < 1$ . Construct a perturbation  $x^1$  of the allocation  $x$  that differs from  $x$  only in the following cases:

$$\begin{aligned} x^1(b_i, s_1) &= x(b_i, s_1) - \eta \\ x^1(b_i, s_{j'}) &= x(b_i, s_{j'}) + \frac{p_s(s_1)}{p_s(s_{j'})}\eta \\ x^1(b_{i'}, s_{j'}) &= x(b_{i'}, s_{j'}) - \frac{p_b(b_i)}{p_b(b_{i'})} \frac{p_s(s_1)}{p_s(s_{j'})}\eta \\ x^1(b_{i'}, s_1) &= x(b_{i'}, s_1) + \frac{p_b(b_i)}{p_b(b_{i'})}\eta. \end{aligned}$$

For sufficiently small  $\eta > 0$ , the inequalities above imply that all the values of  $x^1$  are in  $[0, 1]$ . Moreover,  $x^1$  generates the same allocation probability as  $x$  for each buyer and seller type, and has the property that  $x^1(b_i, s_1) < 1$ . The conclusion of footnote 6 then follows from applying Proposition 4 to allocation  $x^1$ .  $\square$

*Proof of Lemma 3.* We first prove the claim for optimal implementable mechanisms  $(x, t)$  with the property that  $\bar{x}_b$  is strictly increasing and  $\bar{x}_s$  is strictly decreasing. Then, we deal with the more involved case in which these assumptions do not hold.

**Case 1.** Suppose that  $(x, t)$  is an optimal implementable mechanism that is not ex post efficient such that  $\bar{x}_b$  is strictly increasing and  $\bar{x}_s$  is strictly decreasing. Then,  $(x, t)$  must satisfy the constraints from the statement with equality.

Let  $T$  be the set of monetary transfer functions among those that implement  $x$  and maximize the expected utility of a type  $s_n$  seller. As argued in the proof of Lemma 2, every mechanism  $(x, t')$  with  $t' \in T$  satisfies the constraints  $IR_{b_1}$ ,  $IC_{b_i \rightarrow b_{i-1}}$  and  $IC_{s_j \rightarrow s_{j+1}}$  with equality. We will show that for any  $t' \in T$ ,  $(x, t')$  also satisfies  $IR_{s_n}$  with equality. However, if  $IR_{s_n}$  holds with equality at all  $(x, t')$  with  $t' \in T$ , then the maximum utility of a type  $s_n$  seller is zero for all monetary transfer functions that implement  $x$ . Then  $(x, t)$  also achieves the maximum, hence  $t \in T$  and  $(x, t)$  satisfies all the required constraints with equality as well.

Let  $t' \in T$ . The fact that all  $IC_{b_i \rightarrow b_{i-1}}$  and  $IC_{s_j \rightarrow s_{j+1}}$  hold with equality, the strict monotonicity of  $\bar{x}_b$  and  $\bar{x}_s$ , and Lemma 1 imply that all  $IC_{b_{i-1} \rightarrow b_i}$  and  $IC_{s_{j+1} \rightarrow s_j}$  hold with strict inequality for  $(x, t')$ . Suppose that  $IR_{s_n}$  does not hold with equality for  $(x, t')$ . Consider the monetary transfer function  $t''$  defined by

$$t''(b_i, s_j) = t'(b_i, s_j) - (i + j)\varepsilon.$$

By continuity, for  $\varepsilon > 0$  sufficiently small, the mechanism  $(x, t'')$  still satisfies all  $IR_{s_n}$ ,  $IC_{b_{i-1} \rightarrow b_i}$  and  $IC_{s_{j+1} \rightarrow s_j}$  strictly, but also satisfies all  $IC_{b_i \rightarrow b_{i-1}}$ ,  $IC_{s_j \rightarrow s_{j+1}}$ , and  $IR_{b_1}$  strictly.

Hence,  $(x, t'')$  satisfies the sufficient constraints for implementability from Lemma 1 with strict inequality. Then,  $(x', t'')$  also satisfies the sufficient constraints for any small perturbation  $x'$  around  $x$ , so  $(x', t'')$  is implementable for all  $x'$  s.t.  $\sum_{i,j} |x'(b_i, s_j) - x(b_i, s_j)|$  is sufficiently small. If  $x$  is not ex post efficient, there exists such an  $x'$  that generates more welfare, a contradiction with the optimality of  $x$ . This concludes the proof for this case as explained in the first paragraph.

**Case 2.** Suppose that  $(x, t)$  is an optimal mechanism that is not ex post efficient and does not satisfy all the constraints listed in the statement of the lemma with equality. We reach a contradiction.

Since  $x$  is not ex post efficient, either  $\{(i, j) | b_i > s_j, x(b_i, s_j) < 1\}$  or  $\{(i, j) | b_i < s_j, x(b_i, s_j) > 0\}$  is nonempty. We assume that the set  $\{(i, j) | b_i > s_j, x(b_i, s_j) < 1\}$  is not empty and reach a contradiction. A similar contradiction obtains if we assume that the second set is non-empty.

Let  $(i, j) \in \arg \max_{(i,j)} \{i - j | b_i > s_j, x(b_i, s_j) < 1\}$ . Let  $i'$  ( $j'$ ) be the highest (lowest) type of buyer (seller) with  $\bar{x}_b(b_{i'}) = \bar{x}_b(b_i)$  ( $\bar{x}_s(s_{j'}) = \bar{x}_s(s_j)$ ); we have  $i' \geq i, j' \leq j$ . Using arguments similar to those for Case 1, since  $\bar{x}_b(b_{i'}) < \bar{x}_b(b_{i'+1})$  and  $\bar{x}_s(s_{j'-1}) > \bar{x}_s(s_{j'})$ , one can specify the transfer function  $t$  such that  $IR_{b_1}, IR_{s_n}, IC_{b_{i'} \rightarrow b_{i'+1}}, IC_{b_{i'+1} \rightarrow b_{i'}}, IC_{s_{j'} \rightarrow s_{j'-1}}$  and  $IC_{s_{j'-1} \rightarrow s_{j'}}$  hold strictly under  $(x, t)$ . We seek a perturbation of the mechanism  $(x, t)$  that increases  $x(b_i, s_j)$  while preserving the monotonicity of  $\bar{x}_b$  and  $\bar{x}_s$  as in the proof of Proposition 2.

If  $i \neq i'$ , then the definition of  $(i, j)$  implies that  $x(b_i, s_j) < 1 = x(b_{i'}, s_j)$ . Since  $\bar{x}_b(b_{i'}) = \bar{x}_b(b_i)$ , there exists  $j''$  such that  $x(b_i, s_{j''}) > x(b_{i'}, s_{j''})$ . Analogously, if  $j \neq j'$ , then there exists  $i''$  such that  $x(b_{i''}, s_j) > x(b_{i'}, s_j)$ . In particular,  $x(b_i, s_j) < 1, x(b_i, s_{j''}) > 0, x(b_{i'}, s_{j''}) <$

1,  $x(b_{i''}, s_j) > 0$ ,  $x(b_{i''}, s_{j'}) < 1$ . Let  $x'$  be a perturbation of  $x$  for the following types:

$$\begin{aligned} x'(b_i, s_j) &= x(b_i, s_j) + \varepsilon \\ x'(b_i, s_{j''}) &= x(b_i, s_{j''}) - \frac{p_s(s_j)}{p_s(s_{j''})}\varepsilon \\ x'(b_{i'}, s_{j''}) &= x(b_{i'}, s_{j''}) + \frac{p_b(b_i)}{p_b(b_{i'})} \frac{p_s(s_j)}{p_s(s_{j''})}\varepsilon \\ x'(b_{i''}, s_j) &= x(b_{i''}, s_j) - \frac{p_b(b_i)}{p_b(b_{i''})}\varepsilon \\ x'(b_{i''}, s_{j'}) &= x(b_{i''}, s_{j'}) + \frac{p_s(s_j)}{p_s(s_{j'})} \frac{p_b(b_i)}{p_b(b_{i''})}\varepsilon. \end{aligned}$$

If  $i = i'$ , then we set  $i = i' = i''$  and  $x'(b_i, s_j) = x(b_i, s_j) + \varepsilon$  hereafter. We proceed analogously for  $j = j'$ .

The selection of  $i, j, i', j', i'', j''$  guarantees that all values of  $x'$  are in  $[0, 1]$  for small  $\varepsilon$ . Note that  $\bar{x}'_b$  differs from  $\bar{x}_b$  only at  $b_{i'}$ , and  $\bar{x}'_s$  differs from  $\bar{x}_s$  only at  $s_{j'}$ , with the differences computed as follows:

$$\begin{aligned} \bar{x}'_b(b_{i'}) &= \bar{x}_b(b_{i'}) + \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j)\varepsilon \\ \bar{x}'_s(s_{j'}) &= \bar{x}_s(s_{j'}) + \frac{p_s(s_j)}{p_s(s_{j'})} p_b(b_i)\varepsilon. \end{aligned}$$

Since  $IR_{b_1}, IR_{s_n}, IC_{b_{i'} \rightarrow b_{i'+1}}, IC_{b_{i'+1} \rightarrow b_{i'}}, IC_{s_{j'} \rightarrow s_{j'-1}}$  and  $IC_{s_{j'-1} \rightarrow s_{j'}}$  hold strictly under  $(x, t)$ , the only constraints from Lemma 1 that  $(x', t)$  might fail for small  $\varepsilon > 0$  are  $IC_{b_{i'-1} \rightarrow b_{i'}}$  and  $IC_{s_{j'+1} \rightarrow s_{j'}}$ . In order to relax these constraints and implement the allocation  $x'$ , consider a perturbation  $t'$  that coincides with  $t$  with the following exceptions:

$$\begin{aligned} t'(b_{i'}, s_{j'}) &= t(b_{i'}, s_{j'}) + b_{i'} \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j)\varepsilon + s_{j'} \frac{p_s(s_j)}{p_s(s_{j'})} p_b(b_i)\varepsilon \\ t'(b_k, s_{j'}) &= t(b_k, s_{j'}) + s_{j'} \frac{p_s(s_j)}{p_s(s_{j'})} p_b(b_i)\varepsilon, \forall k \neq i' \\ t'(b_{i'}, s_g) &= t(b_{i'}, s_g) + b_{i'} \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j)\varepsilon, \forall g \neq j'. \end{aligned}$$

We have that

$$\begin{aligned} \bar{t}'_b(b_{i'}) &= \bar{t}_b(b_{i'}) + s_{j'} p_s(s_j) p_b(b_i)\varepsilon + b_{i'} \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j)\varepsilon \\ \bar{t}'_s(s_{j'}) &= \bar{t}_s(s_{j'}) + b_{i'} p_b(b_i) p_s(s_j)\varepsilon + s_{j'} \frac{p_s(s_j)}{p_s(s_{j'})} p_b(b_i)\varepsilon \\ \bar{t}'_b(b_k) &= \bar{t}_b(b_k) + s_{j'} p_s(s_j) p_b(b_i)\varepsilon, \forall k \neq i' \\ \bar{t}'_s(s_g) &= \bar{t}_s(s_g) + b_{i'} p_b(b_i) p_s(s_j)\varepsilon, \forall g \neq j'. \end{aligned}$$

By continuity, for small  $\varepsilon > 0$ ,  $IR_{b_1}, IR_{s_n}, IC_{b_{i'} \rightarrow b_{i'+1}}, IC_{b_{i'+1} \rightarrow b_{i'}}, IC_{s_{j'} \rightarrow s_{j'-1}}$  and  $IC_{s_{j'-1} \rightarrow s_{j'}}$  hold strictly under  $(x', t')$ . All constraints that do not involve  $b_{i'}$  or  $s_{j'}$  also hold because at

those values  $\bar{x}'_b$  and  $\bar{x}'_s$  are identical to  $\bar{x}_b$  and  $\bar{x}_s$ , and  $\bar{t}'_b$  and  $\bar{t}'_s$  are translations of  $\bar{t}_b$  and  $\bar{t}_s$ , respectively. The constraints for  $(x', t')$  from Lemma 1 that we are left to check are  $IC_{b_{i'} \rightarrow b_{i'-1}}$ ,  $IC_{b_{i'-1} \rightarrow b_{i'}}$ ,  $IC_{s_{j'} \rightarrow s_{j'+1}}$ , and  $IC_{s_{j'+1} \rightarrow s_{j'}}$ . The constraints  $IC_{b_{i'} \rightarrow b_{i'-1}}$  and  $IC_{b_{i'-1} \rightarrow b_{i'}}$  for  $(x', t')$  are equivalent to the following chain of inequalities

$$b_{i'-1}(\bar{x}'_b(b_{i'}) - \bar{x}'_b(b_{i'-1})) \leq \bar{t}'_b(b_{i'}) - \bar{t}'_b(b_{i'-1}) \leq b_{i'}(\bar{x}'_b(b_{i'}) - \bar{x}'_b(b_{i'-1})),$$

which reduces to

$$\begin{aligned} b_{i'-1}(\bar{x}_b(b_{i'}) - \bar{x}_b(b_{i'-1})) + b_{i'-1} \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j) \varepsilon &\leq \bar{t}_b(b_{i'}) - \bar{t}_b(b_{i'-1}) + b_{i'} \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j) \varepsilon \\ &\leq b_{i'}(\bar{x}_b(b_{i'}) - \bar{x}_b(b_{i'-1})) + b_{i'} \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j) \varepsilon. \end{aligned}$$

The first inequality follows from  $b_{i'-1} < b_{i'}$  and the fact that  $(x, t)$  satisfies  $IC_{b_{i'-1} \rightarrow b_{i'}}$ , while the second inequality is equivalent to  $IC_{b_{i'} \rightarrow b_{i'-1}}$  for  $(x, t)$ . The constraints  $IC_{s_{j'} \rightarrow s_{j'+1}}$  and  $IC_{s_{j'+1} \rightarrow s_{j'}}$  for  $(x', t')$  are checked similarly.

Therefore,  $(x', t')$  is an implementable mechanism. The welfare difference between  $(x', t')$  and  $(x, t)$  depends on the differences in  $\bar{x}'_b$  and  $\bar{x}_b$  at  $b_{i'}$ , and  $\bar{x}'_s$  and  $\bar{x}_s$  at  $s_{j'}$ , and is given by

$$p_b(b_{i'}) b_{i'} \frac{p_b(b_i)}{p_b(b_{i'})} p_s(s_j) \varepsilon - p_s(s_{j'}) s_{j'} \frac{p_s(s_j)}{p_s(s_{j'})} p_b(b_i) \varepsilon = p_b(b_i) p_s(s_j) (b_{i'} - s_{j'}) \varepsilon.$$

As  $b_i > s_j$ ,  $i' \geq i$ , and  $j' \leq j$ , we have that  $b_{i'} > s_{j'}$ , which implies that the expression above is positive. This contradicts the assumed optimality of the mechanism  $(x, t)$ .  $\square$

*Computations for Proposition 4 and footnote 8.* Here we find the optimal implementable mechanisms in the example used to demonstrate Proposition 4 and in an example supporting footnote 8. Assume that the buyer's valuations are  $b_1 = 1$  and  $b_2 = 4$  with probability  $1/2$  each, and that the seller's valuations are  $s_1 = \varepsilon_1$  and  $s_2 = 3 + \varepsilon_2$  with probability  $1/2$  each, where  $\varepsilon_1, \varepsilon_2$  are parameters in  $[0, 1)$ .

We first solve for the optimal mechanism that satisfies the binding constraints from Lemma 3 with equality and then argue that the solution we find is an optimal implementable mechanism. The optimal mechanism with relaxed constraints solves the following linear program:

$$\begin{aligned} \frac{1}{4} \max \quad & (1 - \varepsilon_1)x(b_1, s_1) + (-2 - \varepsilon_2)x(b_1, s_2) + (4 - \varepsilon_1)x(b_2, s_1) + (1 - \varepsilon_2)x(b_2, s_2) + 2(\varepsilon_1 + 3 + \varepsilon_2) \\ \text{s.t.} \quad & x(b_1, s_1) - t(b_1, s_1) + x(b_1, s_2) - t(b_1, s_2) = 0 \\ & t(b_1, s_2) - (3 + \varepsilon_2)x(b_1, s_2) + t(b_2, s_2) - (3 + \varepsilon_2)x(b_2, s_2) = 0 \\ & 4x(b_2, s_1) - t(b_2, s_1) + 4x(b_2, s_2) - t(b_2, s_2) = 4x(b_1, s_1) - t(b_1, s_1) + 4x(b_1, s_2) - t(b_1, s_2) \\ & t(b_1, s_1) - \varepsilon_1 x(b_1, s_1) + t(b_2, s_1) - \varepsilon_1 x(b_2, s_1) = t(b_1, s_2) - \varepsilon_1 x(b_1, s_2) + t(b_2, s_2) - \varepsilon_1 x(b_2, s_2) \\ & x(b_1, s_1), x(b_1, s_2), x(b_2, s_1), x(b_2, s_2) \in [0, 1]. \end{aligned}$$

The equality constraints, correspond in order, to the binding constraints  $IR_{b_1}, IR_{s_2}, IC_{b_2 \rightarrow b_1}$  and  $IC_{s_1 \rightarrow s_2}$  (multiplied by 2).

Rearranging, the system of binding constraints becomes

$$\begin{aligned}
t(b_1, s_1) + t(b_1, s_2) &= x(b_1, s_1) + x(b_1, s_2) \\
t(b_1, s_2) + t(b_2, s_2) &= (3 + \varepsilon_2)(x(b_1, s_2) + x(b_2, s_2)) \\
t(b_1, s_1) + t(b_1, s_2) - t(b_2, s_1) - t(b_2, s_2) &= 4(x(b_1, s_1) + x(b_1, s_2) - x(b_2, s_1) - x(b_2, s_2)) \\
t(b_1, s_1) - t(b_1, s_2) + t(b_2, s_1) - t(b_2, s_2) &= \varepsilon_1(x(b_1, s_1) - x(b_1, s_2) + x(b_2, s_1) - x(b_2, s_2)),
\end{aligned}$$

or

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} t(b_1, s_1) \\ t(b_1, s_2) \\ t(b_2, s_1) \\ t(b_2, s_2) \end{pmatrix} = \begin{pmatrix} x(b_1, s_1) + x(b_1, s_2) \\ (3 + \varepsilon_2)(x(b_1, s_2) + x(b_2, s_2)) \\ 4(x(b_1, s_1) + x(b_1, s_2) - x(b_2, s_1) - x(b_2, s_2)) \\ \varepsilon_1(x(b_1, s_1) - x(b_1, s_2) + x(b_2, s_1) - x(b_2, s_2)) \end{pmatrix}.$$

The linear system above has rank 3, and the vector  $(-2, 2, 1, 1)$  belongs to the left null space of the coefficient matrix. The system has a solution if and only if the rank of the extended matrix is 3 as well, which is equivalent to

$$(-2, 2, 1, 1) \begin{pmatrix} x(b_1, s_1) + x(b_1, s_2) \\ (3 + \varepsilon_2)(x(b_1, s_2) + x(b_2, s_2)) \\ 4(x(b_1, s_1) + x(b_1, s_2) - x(b_2, s_1) - x(b_2, s_2)) \\ \varepsilon_1(x(b_1, s_1) - x(b_1, s_2) + x(b_2, s_1) - x(b_2, s_2)) \end{pmatrix} = 0,$$

or

$$(10) \quad (2 + \varepsilon_1)x(b_1, s_1) + (8 + 2\varepsilon_2 - \varepsilon_1)x(b_1, s_2) + (2 + 2\varepsilon_2 - \varepsilon_1)x(b_2, s_2) = (4 - \varepsilon_1)x(b_2, s_1).$$

The system has a solution  $t(b_1, s_1), t(b_1, s_2), t(b_2, s_1), t(b_2, s_2)$  if and only if (10) is satisfied.

Substituting (10) in the objective function, the optimization problem becomes

$$\begin{aligned}
&\frac{1}{4} \max \quad 3x(b_1, s_1) + (6 + \varepsilon_2 - \varepsilon_1)x(b_1, s_2) + (3 + \varepsilon_2 - \varepsilon_1)x(b_2, s_2) + 2(\varepsilon_1 + 3 + \varepsilon_2) \\
&\text{s.t.} \quad (2 + \varepsilon_1)x(b_1, s_1) + (8 + 2\varepsilon_2 - \varepsilon_1)x(b_1, s_2) + (2 + 2\varepsilon_2 - \varepsilon_1)x(b_2, s_2) = (4 - \varepsilon_1)x(b_2, s_1) \\
&\quad \quad \quad x(b_1, s_1), x(b_1, s_2), x(b_2, s_1), x(b_2, s_2) \in [0, 1].
\end{aligned}$$

At the optimum, we need  $x(b_1, s_2) = 0$ . Indeed, if  $x(b_1, s_2) > 0$  we can increase the value of the objective and satisfy (10) by decreasing  $x(b_1, s_2)$  by a small positive  $\delta$ , and adding  $\delta(8 + 2\varepsilon_2 - \varepsilon_1)/(2 + \varepsilon_1)$  to  $x(b_1, s_1)$  or  $\delta(8 + 2\varepsilon_2 - \varepsilon_1)/(2 + 2\varepsilon_2 - \varepsilon_1)$  to  $x(b_2, s_2)$  (if both  $x(b_1, s_1)$  and  $x(b_2, s_2)$  equal 1, then constraint (10) is violated when  $\varepsilon_1 \geq 0, \varepsilon_2 \geq 0$  with at least one strict inequality). By a similar argument,  $x(b_2, s_1) = 1$ .

The problem reduces to

$$\begin{aligned}
&\frac{1}{4} \max \quad 3x(b_1, s_1) + (3 + \varepsilon_2 - \varepsilon_1)x(b_2, s_2) + 2(\varepsilon_1 + 3 + \varepsilon_2) \\
&\text{s.t.} \quad (2 + \varepsilon_1)x(b_1, s_1) + (2 + 2\varepsilon_2 - \varepsilon_1)x(b_2, s_2) = 4 - \varepsilon_1 \\
&\quad \quad \quad x(b_1, s_1), x(b_2, s_2) \in [0, 1].
\end{aligned}$$

We now specialize the analysis to two examples. For the example used to illustrate Proposition 4, we set  $\varepsilon_2 = 0$ . The unique solution to the linear program above in this case is

$x(b_1, s_1) = 2/(2 + \varepsilon_1), x(b_2, s_2) = 1$ . It is easily checked that the allocation  $x(b_1, s_1) = 2/(2 + \varepsilon_1), x(b_1, s_2) = 0, x(b_2, s_1) = 1, x(b_2, s_2) = 1$  can be implemented with transfers that solve the linear system above. For  $\varepsilon_1 = 0$ ,  $x$  is the ex post efficient allocation, so clearly  $(x, t)$  is an optimal mechanism.

We next argue that  $(x, t)$  is also an optimal implementable mechanism for  $\varepsilon_1 \in (0, 1)$ . It is sufficient to show that the ex post efficient allocation is not implementable in this case. Then, Lemma 3 implies that any optimal implementable mechanism must satisfy the equality constraints we imposed in the linear program, and hence  $(x, t)$  is optimal. For a proof by contradiction, assume that there exists a mechanism  $(x', t')$  that implements the ex post efficient allocation  $x'(b_1, s_1) = 1, x'(b_1, s_2) = 0, x'(b_2, s_1) = 1, x'(b_2, s_2) = 1$  for  $\varepsilon_1 \in (0, 1)$ . Since  $\bar{x}'_b(b_1) = 1/2$  and  $\bar{x}'_b(b_2) = 1$ ,  $IC_{b_2 \rightarrow b_1}$  and  $IR_{b_1}$  imply that the expected payoff of type  $b_2$  under  $(x', t')$  is  $4 - \bar{t}'_b(b_2) \geq 4 \times 1/2 - \bar{t}'_b(b_1) = 3/2 + 1 \times 1/2 - \bar{t}'_b(b_1) \geq 3/2$ . Similarly, the net expected benefit of type  $s_1$  from trading under  $(x', t')$  is  $\bar{t}'_s(s_1) - \varepsilon_1 \times 1 \geq \bar{t}'_s(s_2) - \varepsilon_1 \times 1/2 = 3/2 - \varepsilon_1/2 + \bar{t}'_s(s_2) - 3 \times 1/2 \geq 3/2 - \varepsilon_1/2$ . Hence, the expected gains from trade under  $(x', t')$  should be at least  $1/2(3/2 + 3/2 - \varepsilon_1/2)$ , which is greater than the gains  $x'$  generates,  $1/4((1 - \varepsilon_1) + (4 - \varepsilon_1) + (4 - 3))$ , a contradiction.

For an example corroborating footnote 8, let  $\varepsilon_1 = \varepsilon_2 = \varepsilon$  in the arguments above (a similar conclusion can be reached for cases with  $\varepsilon_1 < \varepsilon_2$ ). The linear program has multiple solutions, one of which is  $x(b_1, s_1) = (2 - 2\varepsilon)/(2 + \varepsilon), x(b_1, s_2) = 0, x(b_2, s_1) = 1, x(b_2, s_2) = 1$ . Arguments similar to those for the first example show there exists an optimal mechanism that implements  $x$  for small  $\varepsilon$ .

The allocation  $x$  achieves a total welfare of  $(24 + 11\varepsilon + 4\varepsilon^2)/(4(2 + \varepsilon))$ , which is decreasing for small  $\varepsilon \geq 0$ . For instance, the welfare generated by  $x$  is 3 for  $\varepsilon = 0$  and approximately 2.993 for  $\varepsilon = 0.1$ . Note that following the damage implied by the decrease in  $\varepsilon$  from 0.1 to 0, the optimal mechanism satisfies the seller's ex ante damage participation constraint described in footnote 8. Indeed, the seller's expected payoff from participating in the post-damage optimal mechanism is 2.25, which is greater than her expected value 1.6 for the undamaged good.  $\square$

*Proof of Proposition 5.* Fix a buyer value distribution  $p_b$ , and consider a seller value distribution  $p'_s$  with support  $s'_1 < s'_2 < \dots < s'_n$  that reflects damage at the top relative to  $p_s$  with support  $s_1 < s_2 < \dots < s_n$ . Let  $(x', t'')$  be an implementable mechanism for the pair of distributions  $(p_b, p'_s)$  that satisfies the seller's damage participation constraint for the value reduction from  $p_s$  to  $p'_s$ . As in the proof of Lemma 2, any implementable mechanism for the pair of distributions  $(p_b, p'_s)$  that implements  $x'$  (not necessarily satisfying the seller's damage participation constraints) and maximizes the expected payoff of seller type  $s'_n$ , satisfies all constraints  $IC_{s_j \rightarrow s_{j+1}}$  with equality. Let  $(x', t')$  be such a mechanism. By definition,  $(x', t')$  generates at least the same utility as  $(x', t'')$  for type  $s'_n$ . Since  $(x', t'')$  obeys the seller's damage participation constraint, it delivers utility of at least  $s_n$  to the seller of type  $s'_n$ , so the same must be true of  $(x', t')$ . It follows that  $(1 - \bar{x}'_s(s'_n))s'_n + \bar{t}'_s(s'_n) \geq s_n$ .

We now construct a mechanism  $(x, t)$  for the pair of distributions  $(p_b, p_s)$  that implements the same type-by-type allocation  $x'$ , i.e.,  $x(b_i, s_j) = x'(b_i, s'_j)$  for all  $i$  and  $j$ , and perturbs the

transfer function  $t'$  as follows:

$$t(b_i, s_j) = t'(b_i, s'_j) - (1 - \bar{x}_s(s_n))(s_n - s'_n) + \sum_{l=j}^{n-1} (\bar{x}_s(s_l) - \bar{x}_s(s_{l+1}))(s_l - s'_l), \forall i = \overline{1, m}, j = \overline{1, n}.$$

Since  $x$  and  $x'$  represent the same allocation of the good for any profile of corresponding types, and seller values under  $p_s$  are higher than under  $p'_s$ , the mechanism  $(x', t')$  with value distributions  $(p_b, p'_s)$  does not achieve greater welfare than the mechanism  $(x, t)$  with value distributions  $(p_b, p_s)$ .

We next argue that  $(x, t)$  is an implementable mechanism for the pair of distributions  $(p_b, p_s)$  by verifying that it satisfies the constraints from Lemma 1. The mechanism  $(x, t)$  satisfies the incentive compatibility constraints for the buyer because the corresponding constraints hold under  $(x', t')$ , and  $\bar{x}_b$  coincides with  $\bar{x}'_b$ , while  $\bar{t}_b(b_i) - \bar{t}'_b(b_i) = \bar{t}_b(b_k) - \bar{t}'_b(b_k)$  for  $i, k = \overline{1, m}$ . The mechanism  $(x, t)$  satisfies  $IR_{b_1}$  because  $(x', t')$  does, and  $t(b_1, s_j) \leq t'(b_1, s_j)$  for  $j = \overline{1, n}$ . The latter inequality holds because

$$\begin{aligned} \sum_{l=j}^{n-1} (\bar{x}_s(s_l) - \bar{x}_s(s_{l+1}))(s_l - s'_l) &\leq \sum_{l=j}^{n-1} (\bar{x}_s(s_l) - \bar{x}_s(s_{l+1}))(s_n - s'_n) \\ &= (\bar{x}_s(s_j) - \bar{x}_s(s_n))(s_n - s'_n) \leq (1 - \bar{x}_s(s_n))(s_n - s'_n), \forall j = \overline{1, n} \end{aligned}$$

as  $s_l - s'_l \leq s_n - s'_n$ ,  $\bar{x}_s(s_l) - \bar{x}_s(s_{l+1}) \geq 0$  for  $l = \overline{j, n-1}$ , and  $\bar{x}_s(s_j) \leq 1$ .

We are left to check that  $(x, t)$  satisfies the constraints  $IR_{s_n}$ ,  $IC_{s_j \rightarrow s_{j+1}}$ , and  $IC_{s_{j+1} \rightarrow s_j}$  for all  $j$ . The constraint  $IR_{s_n}$  holds under  $(x, t)$  since the utility of type  $s_n$  under  $(x, t)$  is

$$(1 - \bar{x}_s(s_n))s_n + \bar{t}(s_n) = (1 - \bar{x}'_s(s'_n))s_n + \bar{t}'_s(s'_n) - (1 - \bar{x}_s(s_n))(s_n - s'_n) = (1 - \bar{x}'_s(s'_n))s'_n + \bar{t}'_s(s'_n) \geq s_n,$$

where the inequality was derived above.

To show that  $(x, t)$  satisfies  $IC_{s_j \rightarrow s_{j+1}}$  and  $IC_{s_{j+1} \rightarrow s_j}$ , note that the fact that  $IC_{s_j \rightarrow s_{j+1}}$  is binding under  $(x', t')$  means that

$$\bar{t}'_s(s'_j) - \bar{t}'_s(s'_{j+1}) = (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s'_j.$$

It follows that

$$\bar{t}_s(s_j) - \bar{t}_s(s_{j+1}) = \bar{t}'_s(s'_j) - \bar{t}'_s(s'_{j+1}) + (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))(s_j - s'_j) = (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_j.$$

Hence,

$$(\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_j = \bar{t}_s(s_j) - \bar{t}_s(s_{j+1}) \leq (\bar{x}_s(s_j) - \bar{x}_s(s_{j+1}))s_{j+1}.$$

This proves that  $(x, t)$  satisfies  $IC_{s_j \rightarrow s_{j+1}}$  and  $IC_{s_{j+1} \rightarrow s_j}$ .

We have argued that  $(x, t)$  is an implementable mechanism for the pair of value distributions  $(p_b, p_s)$  that generates at least the same amount of welfare as  $(x', t')$  for the pair of value distributions  $(p_b, p'_s)$ . Therefore, the optimal implementable mechanism for the value distributions  $(p_b, p_s)$  achieves at least the same amount of welfare as the mechanism  $(x', t')$  for the value distributions  $(p_b, p'_s)$ .  $\square$

*Proof of Proposition 6.* Let  $(x_b, x_s, t_b, t_s)$  be an optimal implementable mechanism. Suppose, for a contradiction, that

$$\Delta := \int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(b, s) - t_s(b, s)) f_b(b) f_s(s) db ds > 0.$$

Perturb transfers as follows:

$$t'_b(b, s) = t'_s(b, s) = t_b(b, s) - \int_{\underline{b}}^{\bar{b}} (t_b(\tilde{b}, s) - t_s(\tilde{b}, s)) f_b(\tilde{b}) d\tilde{b}.$$

Expected transfers under this perturbation are given by

$$\begin{aligned} \bar{t}'_b(b) &= \bar{t}_b(b) - \int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(\tilde{b}, s) - t_s(\tilde{b}, s)) f_b(\tilde{b}) f_s(s) d\tilde{b} ds, \forall b \in [\underline{b}, \bar{b}] \\ \bar{t}'_s(s) &= \bar{t}_s(s), \forall s \in [\underline{s}, \bar{s}]. \end{aligned}$$

Hence, under the perturbation, expected transfers to each seller type are the same as under the original mechanism, while expected transfers from each buyer type are uniformly decreased by  $\int_{\underline{s}}^{\bar{s}} \int_{\underline{b}}^{\bar{b}} (t_b(\tilde{b}, s) - t_s(\tilde{b}, s)) f_b(\tilde{b}) f_s(s) d\tilde{b} ds = \Delta$ . Therefore, the mechanism  $(x_b, x_s, t'_b, t'_s)$  is implementable and yields an increase in welfare of  $\Delta > 0$  over the mechanism  $(x_b, x_s, t_b, t_s)$ , a contradiction with the optimality of the latter.  $\square$

*Proof of Proposition 7.* Let  $(x_b, x_s)$  be an implementable allocation for the pair of regular value distributions  $(F_b, F_s)$ . Let  $U_b(b)$  and  $U_s(s)$  denote the utility the buyer of type  $b$  and the seller of type  $s$  gain from participating in the mechanism (in addition to  $s$  for the seller), respectively. As in Myerson (1981) and Myerson and Satterthwaite (1983), revenue equivalence leads to

$$\begin{aligned} U_b(b) &= U_b(\underline{b}) + \int_{\underline{b}}^b \bar{x}_b(b) f_b(b) db \\ U_s(s) &= U_s(\bar{s}) + \int_s^{\bar{s}} (1 - \bar{x}_s(s)) f_s(s) ds. \end{aligned}$$

Then, the gains from trade  $\int_{\underline{b}}^{\bar{b}} U_b(b) f_b(b) db + \int_{\underline{s}}^{\bar{s}} U_s(s) f_s(s) ds$  achieved by a mechanism implementing  $(x_b, x_s)$  are given by

$$U_b(\underline{b}) + U_s(\bar{s}) + \int_{\underline{b}}^{\bar{b}} (1 - F_b(b)) \bar{x}_b(b) f_b(b) db + \int_{\underline{s}}^{\bar{s}} F_s(s) (1 - \bar{x}_s(s)) f_s(s) ds.$$

Since the gains from trade can be computed also as

$$(11) \quad \int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\bar{s}} (x_b(b, s)b - (1 - x_s(b, s))s) f_b(b) f_s(s) ds db,$$

we get that

$$(12) \quad U_b(\underline{b}) + U_s(\bar{s}) = \int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\bar{s}} \left( x_b(b, s) \left( b - \frac{1 - F_b(b)}{f_b(b)} \right) - (1 - x_s(b, s)) \left( s + \frac{F_s(s)}{f_s(s)} \right) \right) f_b(b) f_s(s) ds db \geq 0.$$

The proof of Theorem 1 in Myerson and Satterthwaite (1983) can be easily adapted to show that an allocation  $(x_b, x_s)$  is implementable if and only if inequality (12) holds,  $x_b(b, s) + x_s(b, s) \leq 1$  for all pairs of types  $(b, s)$ , and  $\bar{x}_b$  and  $\bar{x}_s$  are increasing functions (recall that in our setting  $\bar{x}_s(s)$  represents the probability that seller type  $s$  keeps the good, and not the probability that type  $s$  trades the good, which is the primitive variable in the analysis of Myerson and Satterthwaite; the two variables have opposite monotonicity).

Consider now the relaxed optimization problem of maximizing the gains from trade given by formula (11)—which is equivalent to maximizing total welfare—subject to (12) and  $x_b(b, s) + x_s(b, s) \leq 1$  for all  $(b, s)$ . If  $\lambda \geq 0$  denotes the Lagrange multiplier on constraint (12), we obtain the Lagrangian

$$(\lambda + 1) \int_{\underline{b}}^{\bar{b}} \int_{\underline{s}}^{\bar{s}} \left( x_b(b, s) \left( b - \alpha \frac{1 - F_b(b)}{f_b(b)} \right) - (1 - x_s(b, s)) \left( s + \alpha \frac{F_s(s)}{f_s(s)} \right) \right) f_b(b) f_s(s) ds db,$$

where  $\alpha := \lambda/(\lambda + 1) \in [0, 1)$ .

Let  $c_b(b, \alpha)$  and  $c_s(s, \alpha)$  denote the coefficients multiplying the terms  $x_b(b, s)$  and  $-(1 - x_s(b, s))$  in the integrand above, respectively. Choosing  $x_b(b, s)$  and  $x_s(b, s)$  such that  $x_b(b, s) + x_s(b, s) \leq 1$  to maximize the integrand pointwise for every pair  $(b, s)$  requires that if  $c_b(b, \alpha) > c_s(s, \alpha)$ , then  $x_b(b, s) = 1$  and  $x_s(b, s) = 0$ , and if  $c_b(b, \alpha) < c_s(s, \alpha)$ , then  $x_b(b, s) = 0$  and  $x_s(b, s) = 1$ . Let  $x^\alpha$  be an allocation with these properties.

As in the argument for Theorem 2 of Myerson and Satterthwaite, the assumption that  $F_b$  and  $F_s$  are regular distributions implies that for any  $\alpha \in [0, 1)$ ,  $c_b(\cdot, \alpha)$  and  $c_s(\cdot, \alpha)$  are strictly increasing functions, and hence  $\bar{x}_b^\alpha$  and  $\bar{x}_s^\alpha$  must be increasing. Moreover, when  $[\underline{b}, \bar{b}] \cap [\underline{s}, \bar{s}] \neq \emptyset$ ,<sup>15</sup> there exists a unique  $\alpha^* \in [0, 1)$  for which constraint (12) holds with equality under  $(x_b^{\alpha^*}, x_s^{\alpha^*})$ . It follows that  $(x_b^{\alpha^*}, x_s^{\alpha^*})$  solves the relaxed optimization problem, and given the monotonicity of  $\bar{x}_b^{\alpha^*}$  and  $\bar{x}_s^{\alpha^*}$ , is an optimal implementable allocation. Moreover, any optimal mechanism must implement an allocation that coincides with  $(x_b^{\alpha^*}, x_s^{\alpha^*})$  with probability 1 under  $F_b \times F_s$ .

The proof is completed by noting that  $x_b^{\alpha^*}(b, s) + x_s^{\alpha^*}(b, s) = 1$  for all pairs  $(b, s)$  such that  $c_b(b, \alpha^*) \neq c_s(s, \alpha^*)$ , and that the latter condition holds with probability 1 (since  $c_b(\cdot, \alpha)$  and  $c_s(\cdot, \alpha)$  are strictly increasing functions, for every  $b$  there is at most one  $s$  such that  $c_b(b, \alpha^*) = c_s(s, \alpha^*)$ , and vice versa).  $\square$

*Proof of Proposition 8.* Let  $B$  and  $S$  be random variables with distributions  $F_b$  and  $F_s$ , respectively. Then,  $B' := \alpha(B)$  is a random variable distributed according to  $G_b$ . It is sufficient to show that for any implementable mechanism  $(x', t')$  for the pair of value distributions  $(G_b, F_s)$ , there exists an implementable mechanism  $(x, t)$  for the pair of value distributions  $(F_b, F_s)$  that generates at least the same amount of welfare as  $(x', t')$  under  $(G_b, F_s)$ . Fix such a mechanism  $(x', t')$ . By Theorem 1 of Myerson and Satterthwaite (1983),  $\bar{x}'_b$  and  $\bar{x}'_s$  are increasing functions.

Define  $x(b, s) = x'(\alpha(b), s)$ , so that  $x(B, s) = x'(B', s)$ . We have that

$$\bar{x}_b(B) = \mathbb{E}[x(B, S)|B] = \mathbb{E}[x'(B', S)|B] = \bar{x}'_b(B').$$

<sup>15</sup>The case of non-overlapping supports is trivial.

Since  $\alpha$  and  $\bar{x}'_b$  are increasing functions,  $\bar{x}_b$  must also be increasing. Similarly,  $\bar{x}_s(S) = \bar{x}'_s(S)$  and  $\bar{x}_s$  is decreasing (under the new notation, like in Section 5,  $\bar{x}_s(s)$  denotes the probability that seller type  $s$  trades the good).

As  $(x', t')$  is implementable for the pair of distributions  $(G_b, F_s)$ , Theorem 1 of Myerson and Satterthwaite (1983) implies that

$$\mathbb{E} \left[ \bar{x}'_b(B') \psi_b^{G_b}(B') \right] \geq \mathbb{E} \left[ \bar{x}_s(S) \psi_s^{F_s}(S) \right].$$

Then, the hypothesis that  $\psi_b^{F_b}(B) \geq \psi_b^{G_b}(\alpha(B)) = \psi_b^{G_b}(B')$ , along with  $\bar{x}_b(B) = \bar{x}'_b(B')$ , leads to

$$\mathbb{E} \left[ \bar{x}_b(B) \psi_b^{F_b}(B) \right] \geq \mathbb{E} \left[ \bar{x}_s(S) \psi_s^{F_s}(S) \right].$$

By Theorem 1 of Myerson and Satterthwaite, the inequality above along with the monotonicity of  $\bar{x}_b$  and  $\bar{x}_s$  implies the existence of a transfer function  $t$  such that  $(x, t)$  is an implementable mechanism for the pair of value distributions  $(F_b, F_s)$ .

The difference in the welfare generated by  $(x, t)$  under  $(F_b, F_s)$  and by  $(x', t')$  under  $(G_b, F_s)$  is

$$\mathbb{E} [\bar{x}_b(B)B] - \mathbb{E} [\bar{x}'_b(B')B'] = \mathbb{E} [\bar{x}_b(B)(B - B')] \geq 0,$$

which completes the proof.  $\square$

*Proof of Proposition 9.* Let  $B$  and  $S$  be random variables with distributions  $F_b$  and  $F_s$ , respectively. Then,  $S' := \alpha(S)$  is a random variable distributed according to  $G_s$ . It is sufficient to show that for any implementable mechanism  $(x', t')$  for the pair of value distributions  $(F_b, G_s)$  that obeys the seller's damage participation constraint for the value reduction from  $F_s$  to  $G_s$ , there exists an implementable mechanism  $(x, t)$  for the pair of value distributions  $(F_b, F_s)$  that generates at least the same amount of welfare as  $(x', t')$  under  $(F_b, G_s)$ . Fix such a mechanism  $(x', t')$ , and define  $x(b, s) = x'(b, \alpha(s))$ , so that  $x(b, S) = x'(b, S')$ . Arguments analogous to those for Proposition 8 establish that  $\bar{x}_b(B) = \bar{x}'_b(B)$  and  $\bar{x}_s(S) = \bar{x}'_s(S')$ , and that  $\bar{x}_b$  is increasing and  $\bar{x}_s$  is decreasing.

The fact that  $(x', t')$  obeys the seller's damage participation constraint for the value reduction from  $F_s$  to  $G_s$  implies that the expected gains from trade of the highest value type  $\alpha(\bar{s})$  under  $(x', t')$  are at least  $\bar{s} - \alpha(\bar{s})$ . As  $(x', t')$  is implementable for the pair of distributions  $(F_b, G_s)$ , Theorem 1 of Myerson and Satterthwaite (1983) implies that

$$\mathbb{E} \left[ \bar{x}'_b(B) \psi_b^{F_b}(B) \right] - \mathbb{E} \left[ \bar{x}'_s(S') \psi_s^{G_s}(S') \right] \geq \bar{s} - \alpha(\bar{s}).$$

The hypothesis that  $\psi_s^{F_s}(s) - \psi_s^{G_s}(\alpha(s)) \leq \bar{s} - \alpha(\bar{s})$  for all  $s$  in the support of  $F_s$ , along with  $\bar{s} \geq \alpha(\bar{s})$  and  $\bar{x}_s(S) = \bar{x}'_s(S') \geq 0$ , implies that  $\bar{x}_s(S) \psi_s^{F_s}(S) - \bar{x}'_s(S') \psi_s^{G_s}(S') \leq \bar{s} - \alpha(\bar{s})$ . Therefore,

$$\mathbb{E} \left[ \bar{x}'_s(S') \psi_s^{G_s}(S') \right] - \mathbb{E} \left[ \bar{x}_s(S) \psi_s^{F_s}(S) \right] \geq -(\bar{s} - \alpha(\bar{s})).$$

Adding the two inequalities displayed above, we obtain

$$\mathbb{E} \left[ \bar{x}'_b(B) \psi_b^{F_b}(B) \right] - \mathbb{E} \left[ \bar{x}_s(S) \psi_s^{F_s}(S) \right] \geq 0.$$

By Theorem 1 of Myerson and Satterthwaite, the inequality above along with the monotonicity of  $\bar{x}_b$  and  $\bar{x}_s$  implies the existence of a transfer function  $t$  such that  $(x, t)$  is an implementable mechanism for the pair of value distributions  $(F_b, F_s)$ .

The difference in welfare generated by  $(x, t)$  under  $(F_b, F_s)$  and by  $(x', t')$  under  $(F_b, G_s)$  is  $\mathbb{E}[(1 - \bar{x}_s(S))(S - S')] \geq 0$ .  $\square$

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