Arrow's IIA Condition, May's Axioms, and the Borda Count

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Abstract

We argue that Arrow's (1951) independence of irrelevant alternatives condition (IIA) is unjustifiably stringent. Although, in elections, it has the desirable effect of ruling out spoilers (Candidate A spoils the election for B if B beats C when all voters rank A low, but C beats B when some voters rank A high - - A "siphons" off support from B), it is stronger than necessary for this purpose. Worse, it makes a voting rule insensitive to voters' preference intensities. Accordingly, we propose a modified version of IIA to address these problems. Rather than obtaining an impossibility result, we show that a voting rule satisfies modified IIA, Arrow's other conditions, and May's (1952) axioms for majority rule if and only if it is the Borda count (Borda 1781), i.e., rank-order voting.

1. Arrow, May, and Borda

A. *Arrow's IIA Condition*

In his monograph *Social Choice and Individual Values* (Arrow 1951), Kenneth Arrow introduced the concept of a *social welfare function* (SWF) – a mapping from profiles of individuals' preferences to social preferences.[1] The centerpiece of his analysis was the celebrated

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[1] Formal definitions are provided in section 2.

Impossibility Theorem, which establishes that, with three or more social alternatives, there exists no SWF satisfying four attractive conditions: *unrestricted domain* (U), the *Pareto Principle* (P), *non-dictatorship* (ND), and *independence of irrelevant alternatives* (IIA).

Condition U requires merely that a social welfare function be defined for all possible profiles of individual preferences (since ruling out preferences in advance could be difficult). P is the reasonable requirement that if all individuals (strictly) prefer alternative *x* to *y*, then *x* should be (strictly) preferred to *y* socially as well. ND is the weak assumption that there should not exist a single individual whose strict preference always determines social preference.

These first three conditions are all so undemanding that virtually any SWF studied in theory or used in practice satisfies them all. For example, consider *plurality rule* (or "first-past-the-post"), in which *x* is preferred to *y* socially if the number of individuals ranking *x* first is bigger than the number ranking *y* first.[2] Plurality rule satisfies U because it is well-defined regardless of individuals' preferences. It satisfies P because if all individuals strictly prefer *x* to *y*, then *x* must be ranked first by more individuals than *y*.[3] Finally, it satisfies ND because if everyone else ranks *x* first, then even if the last individual strictly prefers *y* to *x*, *y* will not be ranked above *x* socially.

By contrast, IIA – which requires that social preferences between *x* and *y* should depend only on individuals' preferences between *x* and *y*, and not on preferences concerning some third

[2] As used in elections, plurality rule is strictly speaking a *voting rule*, not a SWF: it merely determines the *winner* (the candidate who is ranked first by a plurality of voters). By contrast, a SWF requires that *all* candidates be ranked socially (Arrow 1951 sees this as a contingency plan: if the top choice turns out not to be feasible, society can move to the second choice, etc.). For most of this paper we will follow the Arrow tradition and concentrate on SWFs. However, in section 5 we show that our main result also holds in the voting rule framework.

[3] This isn't quite accurate, because it is possible that *x* is *never* ranked first. But we can ignore this small qualification.

alternative – is satisfied by few SWFs.[4] Even so, it has a compelling justification: to prevent *spoilers* and vote-splitting in elections.

To understand the issue, consider Scenario 1 (modified from Maskin and Sen 2016). There are three candidates – Donald Trump, Marco Rubio, and John Kasich (the example is inspired by the 2016 Republican primary elections) – and three groups of voters. One group (40%) ranks Trump above Kasich above Rubio; the second (25%) places Rubio over Kasich over Trump; and the third (35%) ranks Kasich above Trump above Rubio (see Figure A).

40%	25%	35%
Trump	Rubio	Kasich
Kasich	Kasich	Trump
Rubio	Trump	Rubio

Figure A: Scenario 1

Many Republican primaries in 2016 used plurality rule; so the winner was the candidate ranked first by more voters than anyone else.[5] As applied to Scenario 1, Trump is the winner with 40% of the first-place rankings. But, in fact, a large majority of voters (60%, i.e., the second and third groups) prefer Kasich to Trump. The only reason why Trump wins in Scenario 1 is that Rubio *spoils* the election for Kasich by siphoning off some of his support;[6] Rubio and Kasich split the first-place votes that don't go to Trump.

[4] One SWF that does satisfy IIA is *majority rule*, in which alternative *x* is socially preferred to *y* if a majority of individuals prefer *x* to *y*. However, unless individuals' preferences are restricted, social preferences with majority rule may cycle (i.e., *x* may be preferred to *y*, *y* preferred to *z*, and yet *z* preferred to *x*), as Condorcet (1785) discovered. In that case, majority rule is not actually a SWF (since its social preferences are intransitive).

[5] In actual plurality rule elections, citizens simply vote for a single candidate rather than rank candidates. But this leads to the same winner as long as citizens vote for their most preferred candidate.

[6] More generally, candidate A spoils the election for B if B beats C when all voters rank A low (i.e., below B and C), but C beats B when some voters rank A high (i.e., above B and C), and the rest rank A low.

An SWF that satisfies IIA avoids spoilers and vote-splitting. To see this, consider Scenario 2, which is the same as Scenario 1 except that voters in the middle group now prefer Kasich to Trump to Rubio (see Figure B).

40%	25%	35%
Trump	Kasich	Kasich
Kasich	Trump	Trump
Rubio	Rubio	Rubio

Figure B: Scenario 2

Pretty much any non-pathological SWF will lead to Kasich being ranked above Trump in Scenario 2 (Kasich is not only top-ranked by 60% of voters, but is ranked second by 40%; by contrast, Trump reverses these numbers: he is ranked first by 40% and second by 60%). However, if the SWF satisfies IIA, it must also rank Kasich over Trump in Scenario 1, since each of the three groups has the same preferences between the two candidates in both scenarios. Hence, unlike plurality rule, a SWF satisfying IIA circumvents spoilers and vote-splitting: Kasich will win in Scenario 1.

But imposing IIA is too demanding: It is stronger than necessary to prevent spoilers (as we will see), and makes sensitivity to preference intensities impossible. To understand this latter point, consider Scenario 3, in which there are three candidates *x*, *y*, and *z* and two groups of voters, one (45% of the electorate) who prefer *x* to *z* to *y*; and the other (55%), who prefer *y* to *x* to *z* (see Figure C).

45%	55%	Under the Borda count
x	*Y*	*x* gets 3 × 45 + 2 × 55 = 245 points
z	*X*	*y* gets 3 × 55 + 1 × 45 = 215 points
y	*Z*	*z* gets 2 × 45 + 1 × 55 = 145 points

so the social ranking is $\begin{matrix} x \\ y \\ z \end{matrix}$

Figure C: Scenario 3

For this scenario, let's apply the *Borda count* (rank-order voting), in which, if there are *m* candidates, a candidate gets *m* points for every voter who ranks her first, $m-1$ points for a second-place ranking, and so on. Candidates are then ranked according to their vote totals. The calculations in Figure C show that in Scenario 3, *x* is socially preferred to *y* and *y* is socially preferred to *z*. But now consider Scenario 4, where the first group's preferences are replaced by *x* over *y* over *z* (see Figure D).

45%	55%	Under the Borda count, the
x	*Y*	
y	*X*	
z	*Z*	social ranking is now $\begin{matrix} y \\ x, \\ z \end{matrix}$ a

violation of IIA as applied to *x* and *y*

Figure D: Scenario 4

As calculated in Figure D, the Borda social ranking becomes *y* over *x* over *z*. This violates IIA: in going from Scenario 3 to 4, no individual's ranking of *x* and *y* changes, yet the social ranking switches from *x* above *y* to *y* above *x*.

However, the anti-spoiler/anti-vote-splitting rationale for IIA doesn't apply to Scenarios 3 and 4. Notice that candidate z doesn't split first-place votes with y in Scenario 3; indeed, she is *never* ranked first. Moreover, her position in group 1 voters' preferences in Scenarios 3 and 4 provides potentially useful information about the intensity of those voters' preferences between x and y. In Scenario 3, z lies between x and y – suggesting that the preference gap between x and y may be substantial. In the second case, z lies below both x and y, implying that the difference between x and y is not as big. Thus, although z may not be a strong candidate herself (i.e., she is, in some sense, an "irrelevant alternative"), how individuals rank her vis à vis x and y is arguably pertinent to social preferences[7],[8] i.e., IIA should not apply to these scenarios.

Accordingly, we propose a relaxation of IIA.[9] Under *modified independence of irrelevant alternatives* (MIIA), if given two alternatives x and y and two profiles of individuals' preferences, (i) each individual ranks x and y the same way in the first profile as in the second,

[7] Here is one way of making the argument formal: Imagine that, from the perspective of an outside observer (society), each of a voter's utilities $u(x)$, $u(y)$, and $u(z)$ (where u captures preference intensity) is drawn randomly and independently from some distribution. Then, the expected difference $u(x) - u(y)$ conditional on z being between x and y in the voter's preference ordering is greater than the difference conditional on z *not* being between x and y.

[8] One might wonder why, instead of depending only on individuals' ordinal rankings, a SWF is not allowed to depend on their *cardinal* utilities, as in Benthamite utilitarianism (Bentham, 1789) or majority judgement (Balinski and Laraki, 2010). But it is not at all clear how to ascertain these utilities, even leaving aside the question of deliberate misrepresentation by individuals. Indeed, for that reason, Lionel Robbins (1932) rejected the idea of cardinal utility altogether, and Arrow (1951) followed in that tradition. Notice that in the case of ordinal preferences, there is an experiment we can perform to verify an individual's ranking: if he says he prefers x to y, we can offer him the choice and see which he selects. But there is no corresponding experiment for cardinal utility - - except in the case of risk preferences, where we can offer lotteries. Yet, risk preferences are not germane to a setting with no risk, and so don't solve the problem at hand. Moreover, even if there were an experiment for eliciting utilities, misrepresentation might interfere with it. Admittedly, there are circumstances when individuals have the incentive to misrepresent their rankings with the Borda count. But a cardinal SWF is subject to much greater misrepresentation because individuals have the incentive to distort even when there are only two alternatives (see Dasgupta and Maskin 2020). Thus, we are left only with the possibility of inferring cardinal qualities – such as preference intensities – from *ordinal* preferences.

[9] Other relaxations of IIA examined in the literature include Saari's (1998) *intensity independence of irrelevant alternatives* (which is strictly stronger than MIIA) and Roberts' (2009) *endogenous independence of irrelevant alternatives* (which considers a variable set of alternatives).

and (ii) each individual ranks the same set of alternatives *between* *x* and *y* in the first profile as in the second, then the social ranking of *x* and *y* must be the same for both profiles.

If we imposed only requirement (i), then MIIA would be identical to IIA. Requirement (ii) is the one that permits preference intensities to figure in social rankings. Specifically, notice that, since *z* lies between *x* and *y* in group 1's preferences in Scenario 3 but not in Scenario 4, MIIA does *not* require the social rankings of *x* and *y* to be the same in the two scenarios. That is, accounting for preference intensities is permissible under MIIA.

Even so, MIIA is strong enough to rule out spoilers and vote-splitting (i.e., a SWF satisfying MIIA cannot exhibit the phenomenon of footnote 6). In particular, it rules out plurality rule: in neither Scenario 1 nor Scenario 2 do group 2 voters rank Rubio between Kasich and Trump. Therefore, MIIA implies that the social ranking of Kasich and Trump must be the *same* in the two scenarios, contradicting plurality rule.

Runoff voting is also ruled out by MIIA. Under that voting rule, a candidate wins immediately if he is ranked first by a majority of voters.[10] But failing that, the two top vote-getters go to a runoff. Notice, that if we change Scenario 1 so that the middle group constitutes 35% of the electorate and the third group constitutes 25%, then Trump (with 40% of the votes) and Rubio (with 35%) go to the runoff (and Kasich, with only 25%, is left out). Trump then wins in the runoff, because a majority of voters prefer him to Rubio. If we change Scenario 2 correspondingly (so that the 25% and 35% groups are interchanged), then Kasich wins in the first round with an outright majority. Thus, runoff voting violates MIIA for the same reason that plurality rule does.

[10] Like plurality rule, runoff voting in *practice* is usually administered so that a voter just picks one candidate rather than ranking them all (see footnote 5).

B. May's Axioms for Majority Rule

When there are just two alternatives, majority rule is far and away the most widely used democratic method for choosing between them. Indeed, almost all other commonly used voting rules – e.g., plurality rule, runoff voting, and the Borda count – reduce to majority rule in this case.

May (1952) crystallized why majority rule is so compelling in the two alternative case by showing that it is the only voting rule satisfying *anonymity* (A), *neutrality* (N), and *positive responsiveness* (PR). Axiom A is the requirement that all individuals be treated equally i.e., that if they exchange preferences with one another (so that individual j gets i's preferences, individual k get j's , and so on), social preferences remain the same. N demands that all alternatives be treated equally i.e., that if the alternatives are permuted and individuals' preferences are changed accordingly, then social preferences are changed in the same way.[11] And PR requires that if alternative x rises relative to y in some individuals' preference orderings, then (i) x doesn't fall relative to y in the social ordering, and (ii) if x and y were previously tied socially, x is now strictly above y.

[11] May expressed this and the PR axioms only for the case of two alternatives. In section 2 we give formal extensions for three or more alternatives (See also Dasgupta and Maskin 2020).

C. The Borda Count and Condorcet Cycles: A Central Special Case

The main result of this paper establishes that a SWF satisfies U, MIIA, A, N, and PR (the other Arrow conditions – P and ND – are redundant) if and only if it is the Borda count.[12] [13] Checking that the Borda count satisfies the five axioms is straightforward.[14]

To illustrate the main idea of the proof in the other direction, let us focus on the case of three alternatives *x*, *y*, and *z* and suppose that *F* is a SWF satisfying the five axioms. We will show that when *F* is restricted to the domain of preferences $\left\{\begin{matrix} x & y & z \\ y, & z, & x \\ z & x & y \end{matrix}\right\}$ (i.e., when we consider only profiles with preferences drawn from this domain[15]), it must coincide with the Borda count. Consider, first, the profile in which 1/3 of individuals have ranking $\begin{matrix} x \\ y \\ z \end{matrix}$; 1/3 have ranking $\begin{matrix} y \\ z \\ x \end{matrix}$; and 1/3 have ranking $\begin{matrix} z \\ x \\ y \end{matrix}$.[16] We claim that the social ranking of *x* and *y* that *F* assigns to this profile is *social indifference*:

$$(1) \qquad \begin{matrix} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{matrix} \xrightarrow{F} x \sim y$$

If (1) doesn't hold, then either

[12] Young (1974) also provides an axiomatization of the Borda count, but his axioms are quite different from ours. Saari (2000) and (2000a) provides a vigorous defense of the Borda count based on its geometric properties.

[13] In section 5, we show that the same result holds for voting rules (formally, called *social choice functions*), once the axioms are suitably reformulated in the voting-rule framework.

[14] To see that the Borda count satisfies MIIA, note that if two profiles satisfy the hypotheses of the condition, then the difference between the number of points a given voter contributes to *x* and the number she contributes to *y* must be the *same* for the two profiles (because the number of alternatives ranked between *x* and *y* is the same). Thus, the difference between the total Borda scores of *x* and *y* – and hence their social rankings – are the same.

[15] From U, *F* is defined for every such profile.

[16] From A, we don't need to worry about which individuals have which preferences.

$$(2)\qquad \begin{array}{ccc} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \end{array}$$

or

$$(3)\qquad \begin{array}{ccc} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}$$

But if (2) holds, then apply permutation σ – with $\sigma(x)=y,\ \sigma(y)=z,$ and $\sigma(z)=x$ – to (2).

From N, we obtain

$$(4)\qquad \begin{array}{ccc} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ y & x & z \\ z & y & x \\ x & z & y \end{array} \xrightarrow{F} \begin{array}{c} y \\ z \end{array}$$

Applying σ to (4) and invoking N, we obtain

$$(5)\qquad \begin{array}{ccc} \underline{1/3} & \underline{1/3} & \underline{1/3} \\ z & y & x \\ x & z & y \\ y & x & z \end{array} \xrightarrow{F} \begin{array}{c} z \\ x \end{array}$$

But the profiles in (2), (4), and (5) are the same except for permutations of individuals' preferences, and so, from A, give rise to the same social ranking under *F*, which in view of (2), (4), and 5 must be

$$\begin{array}{c} x \\ y \\ z \\ x \end{array} \quad ,$$

violating transitivity. The analogous contradiction arises if (3) holds. Hence, (1) must hold after all. From MIIA and (1), we have

$$(6)\qquad \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1/3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} x \sim y\text{, for all } a \geq 0 \text{ and } b \geq 0 \text{ such that } a+b=2/3$$

From PR and (6), we have

$$(7) \qquad \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \end{array}, \text{ where } a+b>2/3, \text{ and } a,b,1-a-b \geq 0,$$

and

$$(8) \qquad \begin{array}{ccc} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & z & y \\ y & x & z \\ z & y & x \end{array} \xrightarrow{F} \begin{array}{c} y \\ x \end{array}, \text{ where } a+b<2/3, \text{ and } a,b,1-a-b \geq 0.$$

But (6), (7), and (8) collectively imply that *x* is socially preferred to *y* if and only if *x*'s Borda score exceeds *y*'s Borda score,[17] i.e., *F* is the Borda count. Q.E.D

The domain $\left\{ \begin{array}{ccc} x & z & y \\ y, & x, & z \\ z & y & x \end{array} \right\}$ is called a Condorcet cycle because, as Condorcet (1785) showed, majority rule may cycle for profiles on this domain (indeed, it cycles for the profile in (1)). This domain is the focus of much of the social choice literature, e.g., Arrow (1951) makes crucial use of Condorcet cycles in the proof of the Impossibility Theorem; Barbie et al (2006) show that it is essentially the unique domain (for three alternatives) on which the Borda count is strategy-proof; and Dasgupta and Maskin (2008) show that no voting rule can satisfy all of P, A, N, and IIA on this domain. One implication of our result in this section is that the Borda count comes closer than any other voting rule to satisfying these four axioms on a Condorcet cycle domain - - it satisfies P, A, and N and captures (through MIAA) the "essence" of IIA.

[17] For example, in (7), *x*'s Borda score is $3a+2b+1-a-b$ and *y*'s Borda score is $3(1-a-b)+2a+b$. Hence *x* is Borda-ranked above *y* if and only if

$$3a+2b+1-a-b > 3(1-a-b)+2a+b,$$

which reduces to

$$a+b > 2/3.$$

2. Formal Model and Definitions

Consider a society consisting of a continuum of individuals[18] (indexed by $i \in [0,1]$) and a finite set of social alternatives X, with $|X| = m$.[19] For each individual *i*, let $\Re_i$ be a set of possible *strict* rankings[20] of *X* for individual *i* and let $\succ_i$ be a typical element of $\Re_i$ ($x \succ_i y$ means that individual *i* prefers alternative *x* to *y*). Then, a *social welfare function* (SWF) *F* is a mapping

$$F : \times_{i \in [0,1]} \Re_i \rightarrow \Re \ ,$$

where $\Re$ is the set of all possible social rankings (here we *do* allow for indifference and the typical element is $\succsim$).

The Arrow conditions for a SWF *F* are:

Unrestricted Domain (U): The SWF must determine social preferences for all possible preferences that individuals might have. Formally, for all $i \in [0,1], \Re_i$ consists of *all* strict orderings of *X*.

Pareto Property (P): If all individuals (strictly) prefer *x* to *y*, then *x* must be strictly socially preferred. Formally, for all profiles $\succ_{\cdot} \in \times \Re_i$ and all $x, y \in X$, if $x \succ_i y$ for all *i*, then $x \succ_F y$, where $\succsim_F = F(\succ_{\cdot})$.

Nondictatorship (ND): There exists no individual who always gets his way in the sense that if he prefers *x* to *y*, then *x* must be socially preferred to *y*, regardless of others' preferences. Formally,

[18] Following Dasgupta and Maskin (2008) and (2020), we assume a continuum to ensure that ties between social alternatives – i.e., social indifference – can occur (this is crucial to our proof technique) but are nongeneric.

[19] $|X|$ is the number of alternatives in *X*.

[20] It simplifies the analysis to rule out the possibility that an individual is indifferent between two alternatives.

there does *not* exist i^* such that for all $\succ_{\cdot} \in \times \Re_i$ and all $x, y \in X$, if $x \succ_{i^*} y$, then $x \succ_F y$, where $\succsim_F = F(\succ_{\cdot})$.

Independence of Irrelevant Alternatives (IIA): Social preferences between *x* and *y* should depend only on individuals' preferences between *x* and *y*, and not on their preferences concerning some third alternative. Formally, for all $\succ_{\cdot}, \succ'_{\cdot} \in \times \Re_i$ and all $x, y \in X$, if, for all *i*, $x \succ_i y \Leftrightarrow x \succ'_i y$, then $\succsim_F$ ranks *x* and *y* the same way that $\succsim'_F$ does, where $\succsim_F = F(\succ_{\cdot})$ and $\succsim'_F = F(\succ'_{\cdot})$.

Because we have argued that IIA is too strong, we are interested in the following relaxation:

Modified IIA: If, given two profiles and two alternatives, each individual (i) ranks the two alternatives the same way in both profiles and (ii) ranks the same set of other alternatives *between* the two alternatives in both profiles,[21] then the social preference between *x* and *y* should be the same for both profiles. Formally, for all $\succ_{\cdot}$, $\succ'_{\cdot} \in \times \Re_i$ and all $x, y \in X$, if, for all *i*, and all $z \in X - \{x, y\}$, $x \succ_i y \Leftrightarrow x \succ'_i y$ and $x \succ_i z \succ_i y \Leftrightarrow x \succ'_i z \succ'_i y$, then $\succsim_F$ and $\succsim'_F$ rank *x* and *y* the same way, where $\succsim_F = F(\succsim_{\cdot})$ and $\succsim'_F = F(\succ'_{\cdot})$.[22]

May (1952) characterizes majority rule axiomatically in the case $|X| = 2$. We will consider natural extensions of his axioms to three or more alternatives:

Anonymity (A): If we permute a preference profile so that individual *j* gets *i*'s preferences, *k* gets *j*'s preferences, etc., then the social ranking remains the same. Formally, fix any (measure-preserving) permutation of society $\pi : [0,1] \to [0,1]$. For any profile $\succ_{\cdot} \in \times \Re_i$, let $\succ_{\cdot}^{\pi}$ be the profile such that, for all *i*, $\succ_i^{\pi} = \succ_{\pi(i)}$. Then $F(\succ_{\cdot}^{\pi}) = F(\succ_{\cdot})$.

[21] The Borda count satisfies a more restrictive version of MIIA that replaces hypothesis (ii) with the premise that the same *number* of alternatives lie between *x* and *y*. In the proof of the Theorem, we show how our weaker axiom together with anonymity and neutrality imply the stronger condition (see footnote 31).

[22] This is similar to, but not quite the same as the definition provided in Maskin (2020).

Neutrality (N): Suppose that we permute the alternatives so that *x* becomes *y, y* becomes *z*, etc., and we change individuals' preferences in the corresponding way. Then, if *x* was socially ranked above *y* originally, now *y* is socially ranked above *z*. Formally, for any permutation $\rho : X \to X$ and any profile $\succ_{\cdot} \in \times \Re_i$, let $\succ_{\cdot}^{\rho}$ be the profile such that, for all $x, y \in X$ and all $i \in [0,1]$, $x \succ_i y \Leftrightarrow \rho(x) \succ_i^{\rho} \rho(y).$ Then, for all $x, y \in X$, $x \succsim_F y \Leftrightarrow \rho(x) \succsim_F^{\rho} \rho(y)$, where $\succsim_F = F(\succ_{\cdot})$ and $\succsim_F^{\rho} = F(\succ_{\cdot}^{\rho})$.

With a continuum of individuals, we can't literally count the number of individuals with a particular preference; we have to work with proportions instead. For that purpose, let μ be Lebesgue measure on $[0,1]$. Given profile $\succ_{\cdot}$, interpret $\mu(\{i | x \succ_i y\})$ as the proportion of individuals who prefer *x* to *y*.[23]

Positive Responsiveness (PR)[24]: If we change individuals' preferences so that alternative *x* moves up and *y* moves down relative to each other and to other alternatives (and no other changes are made), then *x* moves up socially relative to *y* (i.e., if *x* and *y* were previously socially indifferent, *x* is now strictly preferred; if *x* was previously socially preferred to *y*, it remains so). Formally, suppose $\succ_{\cdot}$ and $\succ_{\cdot}'$ are two profiles such that, for some $x, y \in X$ and for all $i \in [0,1]$, $x \succ_i z \Rightarrow x \succ_i' z$, $w \succ_i y \Rightarrow w \succ_i' y$, and $r \succ_i s \Leftrightarrow r \succ_i' s$ for all $z \neq x$, $w \neq y$ and $r, s \in X - \{x, y\}$. Then, if $\mu(\{i | y \succ_i x \text{ and } x \succ_i' y\}) > 0$, we have $x \succsim_F y \Rightarrow x \succ_F' y$, where $\succsim_F = F(\succ_{\cdot})$ and $\succsim_F' = F(\succsim')$.

We can now define the Borda count formally:

[23] To be accurate, we must restrict attention to profiles $\succ_{\cdot}$ for which $\{i | x \succ_i y\}$ is a measurable set.

[24] For a different generalization of PR to more than two alternatives, see Horan, Osborne, and Sanver (2019).

Borda Count: Alternative *x* is socially (weakly) preferred to *y* if and only if *x*'s Borda score (where *x* gets *m* points every time an individual ranks it first, $m-1$ points every time an individual ranks it second, etc.) is (weakly) bigger than *y*'s Borda score. Formally, for all $x, y \in X$ and all profiles $\succ_{\cdot} \in \times \Re_i$,

$$x \succsim^B y \iff \int r_{\succ_i}(x) d\mu(i) \geq \int r_{\succ_i}(y) d\mu(i),$$

where $r_{\succ_i}(x) = \left|\{y \in X \mid x \succ_i y\}\right| + 1$ and $\succsim^B$ is the Borda ranking corresponding to $\succ_{\cdot}$.

3. Main Result

We will establish:

Theorem: SWF *F* satisfies U, MIIA, A, N, and PR if and only if *F* is the Borda count.[25]

Proof:[26] For $|X| = 2$, the result follows from May (1952) (since the Borda count coincides with majority rule for the case of two alternatives).

Suppose $X = \{x, y, z\}$. We have already proved the Theorem for the case when profiles are drawn from a Condorcet cycle domain (see section 1C). We now extend the argument to general profiles. Given profile $\succ_{\cdot}$, let $a_{xy}(\succ_{\cdot})$ be the fraction of individuals who have ranking $\begin{matrix} z \\ x \\ y \end{matrix}$ or $\begin{matrix} x \\ y \\ z \end{matrix}$, i.e., $a_{xy}(\succ_{\cdot}) = \mu\{i \mid z \succ_i x \succ_i y \text{ or } x \succ_i y \succ_i z\}$. Define $a_{yx}(\succ_{\cdot})$ analogously. Also, let

[25] That the Borda count satisfies MIIA was established in footnote 14. That it satisfies the other axioms is obvious.

[26] The Theorem is proved by showing that if *F* satisfies the axioms, its social indifference curves (SICs) must coincide with those of the Borda count. If we simply assume that SICs are linear, the proof in this section suffices. Section 4 extends the argument to the case in which SICs are allowed to be nonlinear and polynomial. Although we conjecture that the Theorem is also true when non-polynomial SICs are permissible, the verification awaits future work.

$a_{xzy}(\succsim.) = \{i | x \succ_i z \succ_i y\}$ and define $a_{yzx}(\succ.)$ analogously. Given proportions α_{xy} and α_{yx}, let $I_3^F(\alpha_{xy}, \alpha_{yx})$ be a proportion α_{xyz} such that if, for profile $\succ.$, $\alpha_{xy} = a_{xy}(\succ.), \alpha_{yx} = a_{yx}(\succsim.)$ and $I_3^F(\alpha_{xy}, \alpha_{yx}) = a_{xzy}(\succ.)$ (implying that $a_{yzx}(\succ.) = 1 - \alpha_{xy} - \alpha_{yx} - I_3^F(\alpha_{xy}, \alpha_{yx})$), then $x \sim_F y$, where $\succsim_F = F(\succ.)$. That is, $(\alpha_{xy}, \alpha_{yx}, I_3^F(\alpha_{xy}, \alpha_{yx}), 1 - \alpha_{xy} - \alpha_{yx} - I_3^F(\alpha_{xy}, \alpha_{yx}))$ is a point in proportion space at which society is indifferent between *x* and *y*. In effect, I_3^F determines the *social indifference curve* for *x* and *y*.

$I_3^F(\alpha_{xy}, \alpha_{yx})$ is well defined: From MIIA, the way that $a_{xy}(\succ.)$ is divided into the fraction having ranking $\begin{matrix} z \\ x \\ y \end{matrix}$ and that having ranking $\begin{matrix} x \\ y \\ z \end{matrix}$ doesn't affect the social ranking of *x* and *y*. Nor does the division of $a_{yx}(\succ.)$ matter. Hence, it is legitimate to write I_3^F as function of α_{xy} and α_{yx}.

Furthermore, if $I_3^F(\alpha_{xy}, \alpha_{yx})$ exists,[27] it is *unique*. To see this, suppose there were two such values α_{xzy} and α'_{xzy} with $\alpha_{xzy} < \alpha'_{xzy}$. Now, going from a profile where the proportions are $(\alpha_{xy}, \alpha_{yx}, \alpha_{xzy}, 1 - \alpha_{xy} - \alpha_{yz} - \alpha_{xzy})$ to one with proportions $(\alpha_{xy}, \alpha_{yx}, \alpha'_{xzy}, 1 - \alpha_{xy} - \alpha_{yz} - \alpha'_{xzy})$ entails *x* rising and *y* is falling in individuals' rankings. Thus if $x \sim_F y$ for the first 4-tuple, then PR implies $x \succ_F y$ for the second 4-tuple, contradicting our assumption that $x \sim_F y$ for the second 4-tuple too.

For the Borda count, we have

[27] Note that for extreme values of α_{xy} and α_{yx}, $I_3^F(\alpha_{xy}, \alpha_{yx})$ *doesn't* exist: If, for example, α_{xy} is near 1, then $x \succ_F y$ regardless of the other proportions.

(9) $$I_3^B(\alpha_{xy},\alpha_{yx}) = (2-3\alpha_{xy}-\alpha_{yx})/4$$ [28]

To show that F is the Borda count, it suffices, in view of PR, to show that

(10) $$I_3^F = I_3^B$$ [29]

Assume for now that I_3^F is *linear* (in section 4, we will show that I_3^F cannot be a higher-degree polynomial):

(11) $$I_3^F(\alpha_{xy},\alpha_{yx}) = B_0 + B_{xy}\alpha_{xy} + B_{yx}\alpha_{yx}$$, for constants B_0, B_{xy}, B_{yx}

Now, from A and N, if $\alpha_{xy} = \alpha_{yx} = \alpha$, then

(12) $$I_3^F(\alpha,\alpha) = 1-2\alpha - I_3^F(\alpha,\alpha)$$

Substituting (11) into (12), we obtain

(13) $$2I_3^F(\alpha,\alpha) = 2B_0 + 2(B_{xy}+B_{yx})\alpha = 1-2\alpha$$

Since (13) holds for all α not too large, we infer that

(14) $$B_0 = 1/2$$

and

(15) $$B_{xy} + B_{yx} = -1$$

Next, consider profile

$$\succ_{\cdot}^{*} = \begin{array}{cccccc} \frac{1/6+c}{} & \frac{1/6+c}{} & \frac{1/6+c}{} & \frac{1/6-c}{} & \frac{1/6-c}{} & \frac{1/6-c}{} \\ x & z & y & z & y & x \\ y & x & z & y & x & z \\ z & y & x & x & z & y \end{array},$$ [30] where c is a sufficiently small constant.

[28] To see that (9) holds, observe that the difference between the Borda scores for x and y is

(*) $(\alpha_{xy}-\alpha_{yx}) + 2(I_3^B - (1-\alpha_{xy}-\alpha_{yx}-I_3^B))$

Alternatives x and y are assumed to be socially indifferent. Hence, we set (*) equal to zero and solve for I_3^B obtaining (9)

[29] To see that this suffices, recall the argument in section 1C.

[30] Here we use U, the unrestricted domain condition.

We claim that

(16) $x \sim_F^* y$, where $\succsim_F^* = F(\succ_\cdot^*)$.

Suppose, to that the contrary, that

(17) $x \succ_F^* y$.

Apply permutation σ (where $\sigma(x) = y,\ \sigma(y) = z,$ and $\sigma(z) = x$) to $\succ_\cdot^*$ twice. From (17) and N, we obtain

(18)
$$\begin{array}{cccccc} 1/6+c & 1/6+c & 1/6+c & 1/6-c & 1/6-c & 1/6-c \\ \hline y & x & z & x & z & y \\ z & y & x & z & y & x \\ x & z & y & y & x & z \end{array} \xrightarrow{F} \begin{array}{c} y \\ z \end{array}$$

and

(19)
$$\begin{array}{cccccc} 1/6+c & 1/6+c & 1/6+c & 1/6-c & 1/6-c & 1/6-c \\ \hline z & y & x & y & x & z \\ x & z & y & x & z & y \\ y & x & z & z & y & x \end{array} \xrightarrow{F} \begin{array}{c} z \\ x \end{array}$$

But the profiles in (18) and (19) are just permutations of the individuals in $\succ_\cdot^*$ and so, from A, (17) – (19) yield the social ranking

$$x \succ_F^* y \succ_F^* z \succ_F^* x,$$

violating transitivity. Hence, (17) can't hold, and similarly $y \succ_F^* x$ can't hold. We conclude that (16) holds. Hence, from $\succ_\cdot^*$

(20) $I_3^F(1/3+2c, 1/3-2c) = 1/6 - c$ for c not too big

From (11), (14), (15), and (20), we have

$$1/2 + B_{xy}(1/3+2c) - (1+B_{xy})(1/3-2c) = 1/6 - c$$

and so

(21) $B_{xy} = -3/4$,

which implies, from (15), that

(22) $$B_{yx} = -1/4.$$

Thus (9), (14), (21), and (22) imply that $I_3^F = I_3^B$, establishing the Theorem for $|X| = 3$.

Next, assume that $X = \{x, y, z, w\}$. Given profile $\succ_{\cdot}$, let

$a_{xy}(\succ_{\cdot}) = \mu(\{i \mid (w \succ_i x \succ_i y \text{ or } z \succ_i x \succ_i y) \text{ and } (x \succ_i y \succ_i w \text{ or } x \succ_i y \succ_i z)\}$,

$a_{xzy}(\succ_{\cdot}) = \mu(\{i \mid w \succ_i x \succ_i z \succ_i y \text{ or } x \succ_i z \succ_i y \succ_i w\})$, $a_{xwy}(\succ_{\cdot}) = \mu(\{z \succ_i x \succ_i w \succ_i y \text{ or }$

$x \succ_i w \succ_i y \succ_i z\})$, and $a_{x\bullet\bullet y}(\succ_{\cdot}) = \mu(\{i \mid x \succ_i z \succ_i y \text{ and } x \succ_i w \succ_i y\})$. Analogously, define

$a_{yx}(\succ_{\cdot}), a_{yzx}(\succ_{\cdot}), a_{ywx}(\succ_{\cdot})$, and $a_{y\bullet\bullet x}(\succ_{\cdot})$. Given proportions $\alpha_{xy}, \alpha_{yx}, \alpha_{xzy}, \alpha_{xwy}, \alpha_{yzx}$ and α_{ywx}, let

$I_4^F(\alpha_{xy}, \alpha_{yx}, \alpha_{xzy}, \alpha_{xwy}, \alpha_{yzx}, \alpha_{ywx})$ be the proportion of individuals *i* with $x \succ_i w \succ_i y$ and

$x \succ_i z \succ_i y$ such that if

$$(\alpha_{xy}, \alpha_{yx}, \alpha_{xzy}, \alpha_{xwz}, \alpha_{yzx}, \alpha_{ywx}, I_4^F, 1 - \alpha_{xy} - \alpha_{yx} - \alpha_{xzy} - \alpha_{xwy} - \alpha_{yzx} - \alpha_{ywx} - I_4^F)$$

$$= (a_{xy}(\succ_{\cdot}), a_{yx}(\succ_{\cdot}), a_{xzy}(\succ_{\cdot}), a_{xwy}(\succ_{\cdot}), a_{yzx}(\succ_{\cdot}), a_{ywx}(\succ_{\cdot}), a_{x\bullet\bullet y}(\succ_{\cdot}), a_{y\bullet\bullet x}(\succ_{\cdot})) \text{ for profile } \succ_{\cdot},$$

then

$$x \sim_F y.$$

Assuming linearity of I_4^F, we have

$$I_4^F(\alpha_{xy}, \alpha_{yx}, \alpha_{xzy}, \alpha_{xwy}, \alpha_{yzx}, \alpha_{ywx})$$

$$= C_0 + C_{xy}\alpha_{xy} + C_{yx}\alpha_{yx} + C_{xzy}\alpha_{xzy} + C_{xwy}\alpha_{xwy} + C_{yzx}\alpha_{yzx} + C_{ywx}\alpha_{ywx},$$

with constants $C_0, C_{xy}, C_{yx}, C_{xzy}, C_{xwy}, C_{yzx}, C_{ywx}$. But from A and N, $C_{xzy} = C_{xwy} (= C_{x\bullet y})$ and $C_{yzx} = C_{ywx} (= C_{y\bullet x})$,[31] and so we can take $\alpha_{x\bullet y} = \alpha_{xzy} + \alpha_{xwy}$ and $\alpha_{y\bullet x} = \alpha_{yzx} + \alpha_{ywx}$ and rewrite I_4^F as

(23) $$I_4^F(\alpha_{xy}, \alpha_{yx}, \alpha_{x\bullet y}, \alpha_{y\bullet x}) = C_0 + C_{xy}\alpha_{xy} + C_{yx}\alpha_{yx} + C_{x\bullet y}\alpha_{x\bullet y} + C_{y\bullet x}\alpha_{y\bullet x}$$

Now, if $\alpha_{xy} = \alpha_{yx} = \alpha_1$ and $\alpha_{x\bullet y} = \alpha_{y\bullet x} = \alpha_2$ then from N,

$$I_4^F(\alpha_1, \alpha_1, \alpha_2, \alpha_2) = 1 - 2\alpha_1 - 2\alpha_2 - I_4^F(\alpha_1, \alpha_1, \alpha_2, \alpha_2),$$

i.e.,

(24) $$I_4^F(\alpha_1, \alpha_1, \alpha_2, \alpha_2) = (1 - 2\alpha_1 - 2\alpha_2)/2$$

Hence, from (23) and (24), we have

$$C_0 + (C_{xy} + C_{yx})\alpha_1 + (C_{x\bullet y} + C_{y\bullet x})\alpha_2 = (1 - 2\alpha_1 - 2\alpha_2)/2,$$

and so

(25) $$C_0 = 1/2$$

(26) $$C_{xy} + C_{yx} = -1$$

and

(27) $$C_{x\bullet y} + C_{y\bullet x} = -1$$

Now, by definition of I_3^F and I_4^F,

(28) $$I_4^F(\alpha_{xy}, \alpha_{yx}, I_3^F(\alpha_{xy}, \alpha_{yx}), 1 - \alpha_{xy} - \alpha_{yx} - I_3^F(\alpha_{xy}, \alpha_{yx})) = 0$$

Because $I_3^F(\alpha_{xy}, \alpha_{yx}) = I_3^B(\alpha_{xy}, \alpha_{yx}) = (2 - 3\alpha_{xy} - \alpha_{yx})/4$ we can rewrite (28) using (23) as

[31] Here is where we establish that MIIA together with I and A imply the stronger form of MIIA discussed in footnote 21.

(29) $$C_0 + C_{xy}\alpha_{xy} + C_{yx}\alpha_{yx} + C_{x\bullet y}(2-3\alpha_{xy}-\alpha_{yx})/4 + C_{y\bullet x}(1-\alpha_{xy}-\alpha_{yx}-(2-3\alpha_{xy}-\alpha_{yx})/4) = 0$$

From (29), we obtain

(30) $$4C_{xy} - 3C_{x\bullet y} - C_{y\bullet x} = 0$$

Finally, consider the profile $\succ_{\circ}^{**}$ such that

(31)

$1/24+c_1$	$1/24+c_1$	$1/24+c_1$	$1/24+c_1$
x	y	z	w
y	z	w	x
z	w	x	y
w	x	y	z

(32)

$1/24+c_2$	$1/24+c_2$	$1/24+c_2$	$1/24+c_2$
x	y	w	z
y	w	z	x
w	z	x	y
z	x	y	w

(33)

$1/24+c_3$	$1/24+c_3$	$1/24+c_3$	$1/24+c_3$
x	w	y	z
w	y	z	x
y	z	x	w
z	x	w	y

(34)

$1/24+c_4$	$1/24+c_4$	$1/24+c_4$	$1/24+c_4$
x	z	y	w
z	y	w	x
y	w	x	z
w	x	z	y

(35)

$1/24+c_5$	$1/24+c_5$	$1/24+c_5$	$1/24+c_5$
x	z	w	y
z	w	y	x
w	y	x	z
y	x	z	w

(36)

$1/24-\sum_{j=1}^{5} c_j$	$1/24-\sum c_j$	$1/24-\sum c_j$	$1/24-\sum c_j$
x	w	z	y
w	z	y	x
z	y	x	w
y	x	w	z

From the same argument we used to establish that $x \sim_F^{*} y$ in (16) above, we can show that

(37) $$x \sim_F^{**} y, \text{ where } \succsim^{**} = F(\succ_{\cdot}^{**})$$

From (23) and (31) – (37),

(38) $$\begin{aligned} &C_0 + C_{xy}(1/4 + 3c_1 + 3c_2) + C_{yx}(1/4 - 3c_1 - 3c_2 - 3c_3 - 3c_4) + C_{x\bullet y}(1/6 + 2c_3 + 2c_4) \\ &+ C_{y\bullet x}(1/6 + 2c_3 + 2c_4) \;=\; 1/12 - c_1 - c_2 - c_3 - c_4 \end{aligned}$$

By varying c_1 in (38), we obtain

(39) $$3C_{xy} - 3C_{yx} = -1$$

From (25) – (27), (30), and (39), we obtain

$$C_{xy} = -2/3,\; C_{yx} = -1/3,\; C_{x\bullet y} = -5/6,\; C_{y\bullet x} = -1/6$$

and so

$$I_4^F = I_4^B$$ [32]

Now, we can proceed inductively on *m*. In the same way that we used the fact that $I_3^F = I_3^B$ to show that $I_4^F = I_4^B$, we can show that $I_m^F = I_m^B \Rightarrow I_{m+1}^F = I_{m+1}^B$. Specifically, we can use the following relations:

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[32] We can solve for I_4^B from the following equation:

$$(\alpha_{xy} - \alpha_{yx}) + 2(\alpha_{x\bullet y} - \alpha_{y\bullet x}) + 3(I_4^B - (1 - \alpha_{xy} - \alpha_{yx} - \alpha_{x\bullet y} - \alpha_{y\bullet x} - I_4^B)) = 0$$

$$I_{m+1}^F(\alpha_{xy},\alpha_{yx},\ldots,\alpha_{x\underset{\substack{m-2\\ \text{dots}}}{\cdots\cdots}y},\alpha_{y\cdots\cdots x},I_m^F(\alpha_{xy},\alpha_{yx},\ldots,\alpha_{x\cdots\cdots y},\alpha_{y\cdots\cdots x}),$$

$$1-\alpha_{xy}-\alpha_{yx}-\cdots-\alpha_{x\cdots\cdots y}-\alpha_{y\cdots\cdots x}-I_m^F(\alpha_{xy},\alpha_{yx},\ldots,\alpha_{x\cdots\cdots y},\alpha_{y\cdots\cdots x}))=0,$$

$$I_{m+1}^F(\alpha_1,\alpha_1,\alpha_2,\alpha_2,\ldots,\alpha_{m-1},\alpha_{m-1})$$
$$=(1-2\alpha_1-2\alpha_2-\cdots-2\alpha_{m-1})/2,$$

and

$$I_{m+1}^F\left(m(m-1)!/(m+1)!+m\sum_{j=1}^{(m-1)!}c_j,\ m(m-1)!/(m+1)!-m\sum_{j=1}^{(m-1)(m-1)!}c_j,\ \ldots,\right.$$
$$\left.2(m-1)!/(m+1)!+2\sum_{j=(m-2)(m-1)!+1}^{(m-1)(m-1)!}c_j\right)$$
$$=(m-1)!/(m+1)!-\sum_{j=1}^{(m-1)(m-1)!}c_j$$

to solve for the coefficients of I_{m+1}^F and verify that they are the same as those for I_{m+1}^B.

Q.E.D.

4. Linearity of Social Indifference Curves

Theorem: If I_m^F is a polynomial, then I_m^F is linear

Proof:[33] Consider the case $m=3$. Rather than working with I_3^F, it will be easier to work with

$$J_3^F(\alpha_{xy},\alpha_{xzy})=\alpha_{yx},$$

where $(\alpha_{xy},\alpha_{yx},\alpha_{xzy},1-\alpha_{xy}-\alpha_{yx}-\alpha_{xzy})=(a_{xy}(\succ_.),a_{yx}(\succ_.),a_{xzy}(\succ_.),a_{yzx}(\succ_.))$ and $\succ_.$ is a profile such that $x\sim_{F(\succ_.)}y$. We want to show that if J_3^F is a polynomial, it is linear. Suppose to the contrary that it is quadratic:

[33] An example by Hervé Moulin shows that the techniques used in this proof cannot immediately be extended to the case where I_m^F is not a polynomial.

(40) $$J_3^F(\alpha_{xy},\alpha_{xzy}) = D_0 + D_{xy}\alpha_{xy} + D_{xzy}\alpha_{xzy} + D_{(xy)^2}\alpha_{xy}^2 + D_{(xy)(xzy)}\alpha_{xy}\alpha_{xzy} + D_{(xzy)^2}\alpha_{xzy}^2$$

From N,

(41) $$J_3^F(J_3^F(\alpha_{xy},\alpha_{xzy}), 1-\alpha_{xy}-\alpha_{xzy}-J_3^F(\alpha_{xy},\alpha_{xzy})) = \alpha_{xy}$$

If we expand the left-hand side of (41) using (40), we obtain a quartic expression. And from (41), all coefficients of nonlinear terms must be zero. Hence,

(42) (a) $$D_{(xy)^2}^2(D_{(xy)^2} - D_{(xy)(xzy)} + D_{(xzy)^2}) = 0,$$

(b) $$D_{(xzy)^2}^2(D_{(xy)^2} - D_{(xy)(xzy)} + D_{(xzy)^2}) = 0,$$

where the left-hand sides of (42a) and (42b) are the coefficients of α_{xy}^4 and α_{xzy}^4 respectively.

From the coefficient of α_{xy}^3, (42a), and (42b), we obtain

(43) (a) $$D_{(xy)^2}(D_{(xy)(xzy)} - 2D_{(xzy)^2}) = 0$$

(b) $$D_{(xzy)^2}(D_{(xy)(xzy)} - 2D_{(xzy)^2}) = 0$$

From the coefficient of α_{xy}^2, (42a), and (43a),

we have

(44) $$D_{(xy)^2}(D_{xy} - D_{xzy}) = 0$$

From N,

(45) $$\begin{aligned} & J_3^F(t,\tfrac{1}{2}-t) \\ & = D_0 + D_{xy}t + D_{xzy}(\tfrac{1}{2}-t) + D_{(xy)^2}t^2 + D_{(xy)(xzy)}t(\tfrac{1}{2}-t) + D_{(xzy)^2}(\tfrac{1}{2}-t)^2 \\ & = t, \text{ for all } t \in [0,\tfrac{1}{2}]. \end{aligned}$$

From (45) and the coefficient of t,

(46) $$D_{xy} - D_{xzy} + \frac{1}{2} D_{(xy)(xzy)} - D_{(xzy)^2} = 1$$

Now, if

(47) $$D_{(xy)^2} \neq 0,$$

then, from (44),

(48) $$D_{xy} - D_{xzy} = 0$$

and, from (43a),

(49) $$D_{(xy)(xzy)} - 2D_{(xzy)^2} = 0$$

From (48) and (49), the left-hand side of (46) is zero, a contradiction. Hence

(50) $$D_{(xy)^2} = 0$$

Now, if $D_{(xzy)^2} \neq 0,$

then, from (42b), (43b), and (49)

$$D_{(xy)(xzy)} = D_{(xzy)^2} \text{ and } D_{(xy)(xzy)} = 2D_{(xzy)^2} ,$$

which is impossible. Hence

(51) $$D_{(xzy)^2} = 0$$

From (50) and (51), the coefficient of $\alpha_{xy}^2 \alpha_{xzy}^2$ in the expansion of the left-hand side of (41) is $-D^3_{(xy)(xzy)}$, and so, from (41),

(52) $$D_{(xy)(xzy)} = 0$$

Hence, (50), (51) and (52) imply that J_3^F must be linear. The argument for higher-degree polynomials follows exactly the same lines, and going from $m = 3$ to $m = 4$ (and from m to $m+1$) mimics the argument in the proof of the main theorem. Q.E.D.

5. Voting Rules

Let us now turn to the case in which individuals' preferences are aggregated to produce a social *winner*, rather than a social ranking. That is, we are now concerned with voting rules rather than SWFs. In the social choice literature, voting rules are usually called *social choice functions* (SCFs). Formally, a SCF f is a correspondence

$$f : \times_{i \in [0,1]} \Re_i \longrightarrow\!\!\!\!> X.$$

That is, for all profiles $\succ_\cdot$, $f(\succ_\cdot) \subseteq X$. We can interpret $f(\succ_\cdot)$ as the alternatives that "win" or are "socially optimal" given profile $\succ_\cdot$. We now need to adapt the axioms of section 2 to the SCF framework. For U, A, and N, the adaptation is immediate:

Unrestricted Domain* (U*): For all $i \in [0,1]$, $\Re_i$ consists of all strict orderings of X.

Anonymity* (A*): Fix (measure-preserving) permutation $\pi : [0,1] \to [0,1]$. For any profile $\succ_\cdot \in \times \Re_i$, let $\succ_\cdot^\pi$ be the profile such that, for all i, $\succ_i^\pi = \succ_{\pi(i)}$. Then $f(\succ_\cdot^\pi) = f(\succ_\cdot)$.

Neutrality* (N*): For any permutation $\rho : X \to X$ and any profile $\succ_\cdot \in \times \Re_i$, let $\succ_\cdot^\rho$ be the profile such that, for all $x, y \in X$ and all $i \in [0,1]$, $x \succ_i y \Leftrightarrow \rho(x) \succ_i^\rho \rho(y)$. Then, $\rho(f(\succ_\cdot) = f(\succ_\cdot^\rho)$.

Reformulating MIIA is slightly trickier. One thing we *can't* do is posit that if two profiles $\succ_\cdot$ and $\succ_\cdot'$ satisfy the hypotheses of the condition, then $x, y \in f(\succ_\cdot)$ if and only if $x, y \in f(\succ_\cdot')$, since even if x and y are chosen by f for $\succ_\cdot$, some other alternative z might be preferred by most individuals in profile $\succ_\cdot'$, leading to $\{z\} = f(\succ_\cdot')$. Thus, the " right" translation of MIIA says that if x is chosen by f for $\succ_\cdot$ and y is chosen for $\succ_\cdot'$, then they are *both* chosen for *both* profiles:

*Modified Independence of Irrelevant Alternatives** (MIIA*): For $\succ_\cdot, \ \succ'_\cdot \in \times\Re_i$ and all *x*, *y*,

suppose that, for all *i* and all $z \in X - \{x, y\}$, $x \succ_i y \Leftrightarrow x \succ'_i y$ and $x \succ_i z \succ_i y \Leftrightarrow x \succ'_i z \succ_i y$. Then,

$x \in f(\succ_\cdot)$ and $y \in f(\succ'_\cdot) \Leftrightarrow y \in f(\succ_\cdot)$ and $x \in f(\succ'_\cdot)$.

Finally, the new version of PR says that if, in going from $\succ_\cdot$ to $\succ'_\cdot$, *x* moves up and *y* moves down relative to other alternatives (and no other changes are made), then if *x* was chosen for $\succsim_\cdot$, it is chosen for $\succ'_\cdot$; and if *x* and *y* are both chosen, for $\succ_\cdot$, only *x* is chosen for $\succ'_\cdot$:

*Positive Responsiveness** (PR*): Suppose $\succ_\cdot$ and $\succ'_\cdot$ are two profile such that, for some $x, y \in X$ and for all $i \in [0,1]$, $x \succ_i z \Rightarrow x \succ'_i z$, $w \succ_i y \Rightarrow w \succ'_i y$ and $r \succ_i s \Leftrightarrow r \succ'_i s$ for all $z \neq x, w \neq y$ and $r, s \in X - \{x, y\}$. Then, if $\mu(\{i \mid y \succ_i x \text{ and } x \succ'_i y\}) > 0$, we have $x \in f(\succ_\cdot) \Rightarrow x \in f(\succ'_\cdot)$ and $x, y \in f(\succ_\cdot) \Rightarrow y \notin f(\succ'_\cdot)$

The formal definition of the Borda SCF f^B is:

Borda SCF: f^B is defined as

$$f^B(\succ_\cdot) = \{x \mid \int r_{\succ_i}(x) d\mu(i) \geq \int r_{\succ_i}(y) d\mu(i) \text{ for all } y\}.$$

We can now state the counterpart of our main result in Section 3:

*Theorem**: SCF *f* satisfies U*, MIIA*, A*, N*, and PR* if and only if *f* is the Borda SCF f^B.

Sketch of Proof: To understand why the Theorem* holds, let's return to the scenario of Section 1C, where $X = \{x, y, z\}$ and the domain of preferences is $\left\{\begin{matrix} x & y & z \\ y, & z, & x \\ z & x & y \end{matrix}\right\}$. Consider profile

$$(53) \qquad \begin{matrix} \underline{a} & \underline{b} & \underline{1-a-b} \\ x & z & y \\ y & x & z \\ z & y & x \end{matrix}$$

It is easy to verify that the boundary between the regions of *a*-*b* space in which *x* and *y* are the unique Borda winners is given by

(54) $a+b=2/3,$ provided $b \leq 1/3$ (see Figure E).[34]

Similarly, the boundary between the region where *y* and *z* are the Borda winners is given by

(55) $b=1/3,$ provided $a \leq 1/3$.

Finally, the boundary between the region where *z* and *x* are the Borda winners is given by

(56) $a=1/3,$ provided $b \geq 1/3$.

From PR*, it will suffice to show that if *f* satisfies the axioms, then it has the same boundaries as f^B. Consider the Condorcet paradox profile

$$\succ^* = \begin{array}{ccc} 1/3 & 1/3 & 1/3 \\ \hline x & z & y \\ y & x & z \\ z & y & x \end{array}$$

From A and N,

(57) $\{x,y,z\} = f(\succ^*)$

Now, starting from $(a,b)=(1/3,1/3)$, consider increasing *a* and decreasing *b* in profile (53) to keep $a+b=2/3.$ From PR* and (57), *z* is no longer chosen, and from MIIA*, both *x* and *y are* chosen. Hence, *f* gives rise to the boundary (54), just as f^B does. The argument is analogous for the other two boundaries. Q.E.D.

[34] *x* and *y* are both chosen by f^B if $3a+2b+(1-a-b)=3(1-a-b)+2a+b$, i.e., if $a+b=2/3$, and if in addition, *z*'s Borda score is *not* higher than *y*'s: $3(1-a-b)+2a+3b \geq 3b+2(1-a-b)+a$, i.e., $b \leq 1/3$.

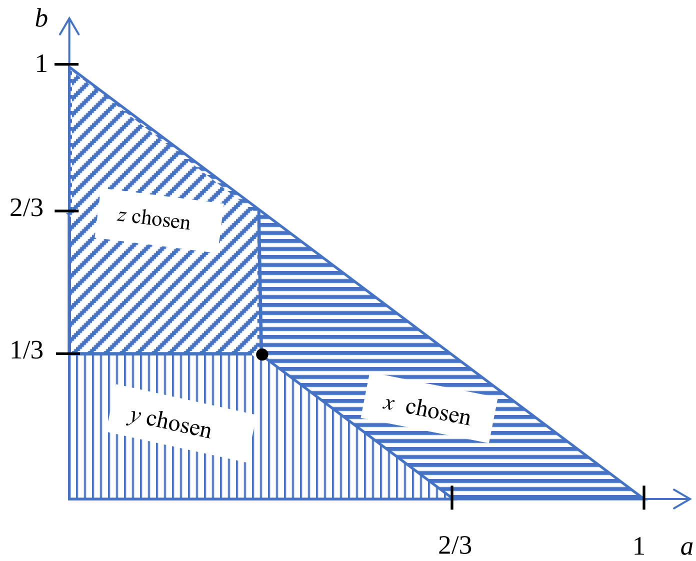

Figure E. Borda SCF

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